ECDLP course

Daniel J. Bernstein
University of Illinois at Chicago

Tanja Lange
Technische Universiteit Eindhoven

Main goal of this course:
We are the attackers.
We want to break ECC.

Enemy: ECC users.
Helpful for us as attackers:
Understand the enemy!

What are ECC users doing?
Why are they doing this?
What will they do tomorrow?

(This perspective also helps
if you want to use ECC.)

Can break ECC in many ways.
But most of these ways can be
easily stopped by ECC users.

Most important strategy
for attacking ECC:
Break ECDLP
by parallel Pollard rho
with all available refinements.

ECC users cannot stop rho,
although they can make it slower
by increasing ECC key size.

Why ECC?

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Many users of 1024-bit RSA:
EFF SSL observatory study
http://eff.org/observatory
shows that more than 50% of all certificates are $\leq$ 1024-bit RSA.

1000 cores in perspective:

Typical laptop has 2 cores.

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The Conficker/Downadup criminal-controlled botnet has $\approx 10\,000\,000$ cores.

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An attacker building ASICs
for 10 million USD can break
RSA-1024 in a year.

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These recommendations
don’t even take into account
batch algorithms
that save time in breaking
many keys together.

A 1024-bit RSA key is built from two secret 512-bit primes.

There are $\approx 2^{503}$
possible 512-bit primes.
Can't imagine trying them all.
But the attacks are much faster:
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Still below modern standards!
Attacks: $\approx 2^{112}$ calculations.

3072-bit key: 1536-bit primes;
$\approx 2^{1526}$ possible primes.
Attacks: $\approx 2^{128}$ calculations.

Attacks use "index calculus"
= "combining congruences."
Long history, including
many major improvements:
1975, CFRAC;
1977, linear sieve (LS);
1982, quadratic sieve (QS);
1990, number-field sieve (NFS).
Also many smaller improvements.

Costs of these algorithms for
breaking RSA-1024, RSA-2048:
$\approx 2^{120}$, 2^{170}, CFRAC;
$\approx 2^{110}$, 2^{160}, LS;
$\approx 2^{100}$, 2^{150}, QS;
$\approx 2^{80}$, 2^{112}, NFS.

1977: RSA is introduced.

1985: Miller proposes ECC.
Explains several obstacles
to congruence-combination
attacks on elliptic curves.

Subsequent ECC history:
Improved algorithms, but
still take exponential time.

Subsequent RSA history:
Continued security losses
from improved algorithms
for combining congruences.
Major loss in 1990 (NFS);
many smaller losses since then.

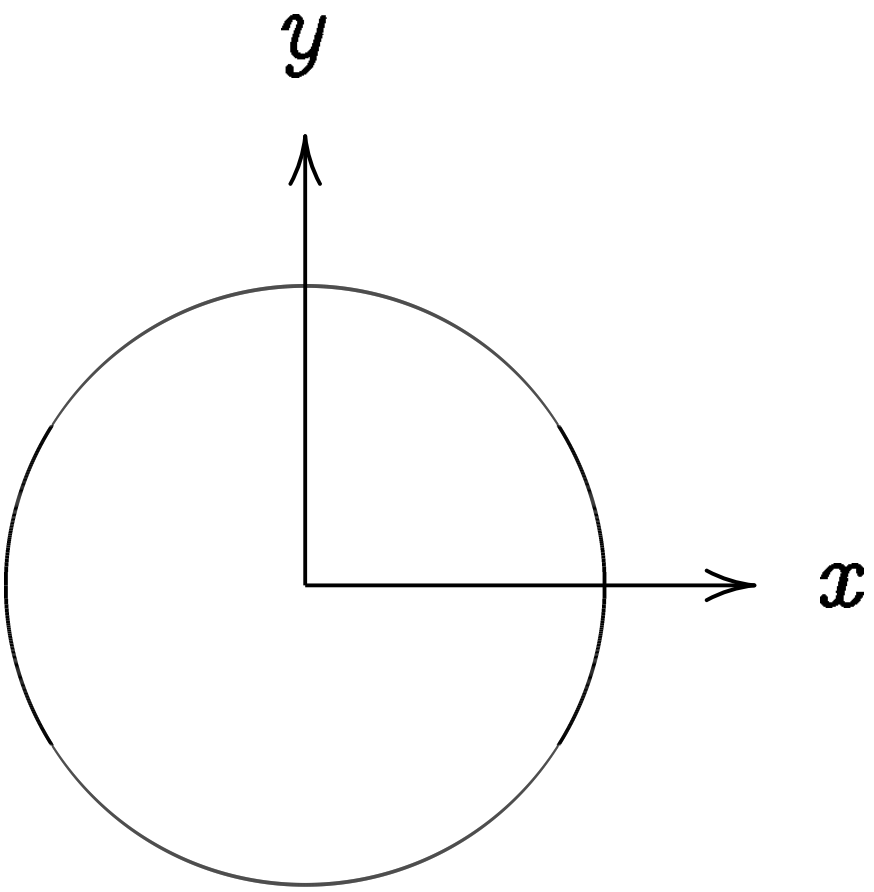
256-bit ECC keys match
security of 3072-bit RSA keys.

When properly implemented,
256-bit ECC is much faster
than 3072-bit RSA for
almost all real-world applications.

ANSI, IEEE, NIST issued
ECC standards ten years ago.
US government “Suite B”
now prohibits RSA, requires ECC.

Electronic ID or travel documents
in many European countries based
on ECC. Some internet protocols
(DNSCurve, CurveCP) use ECC.

The clock



This is the curve $x^2 + y^2 = 1$.

Warning:

This is *not* an elliptic curve.

"Elliptic curve" $\neq$ "ellipse."

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$(-1/2, -\sqrt{3/4}) =$ “7:00”.

$(\sqrt{1/2}, \sqrt{1/2}) =$ “1:30”.

$(3/5, 4/5)$. $(-3/5, 4/5)$.

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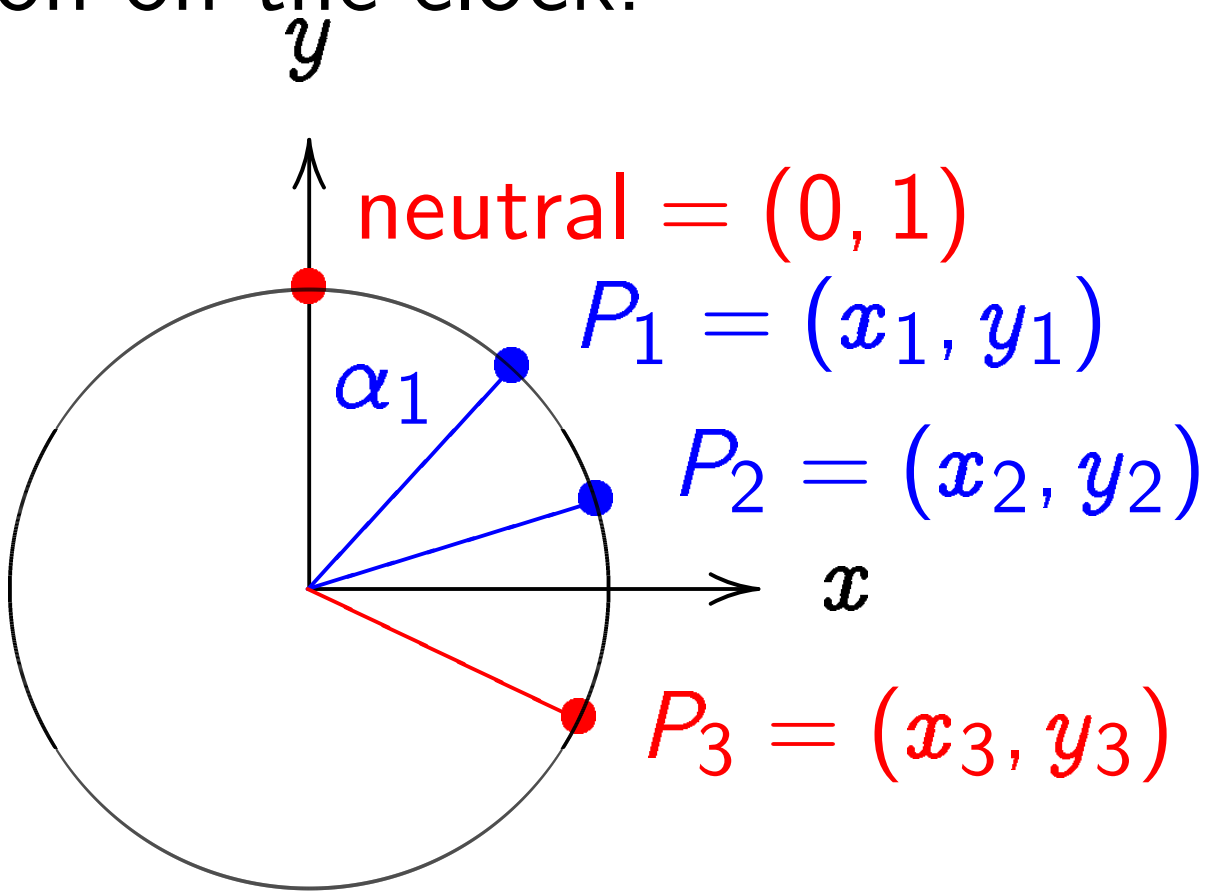
Many more.

Addition on the clock:



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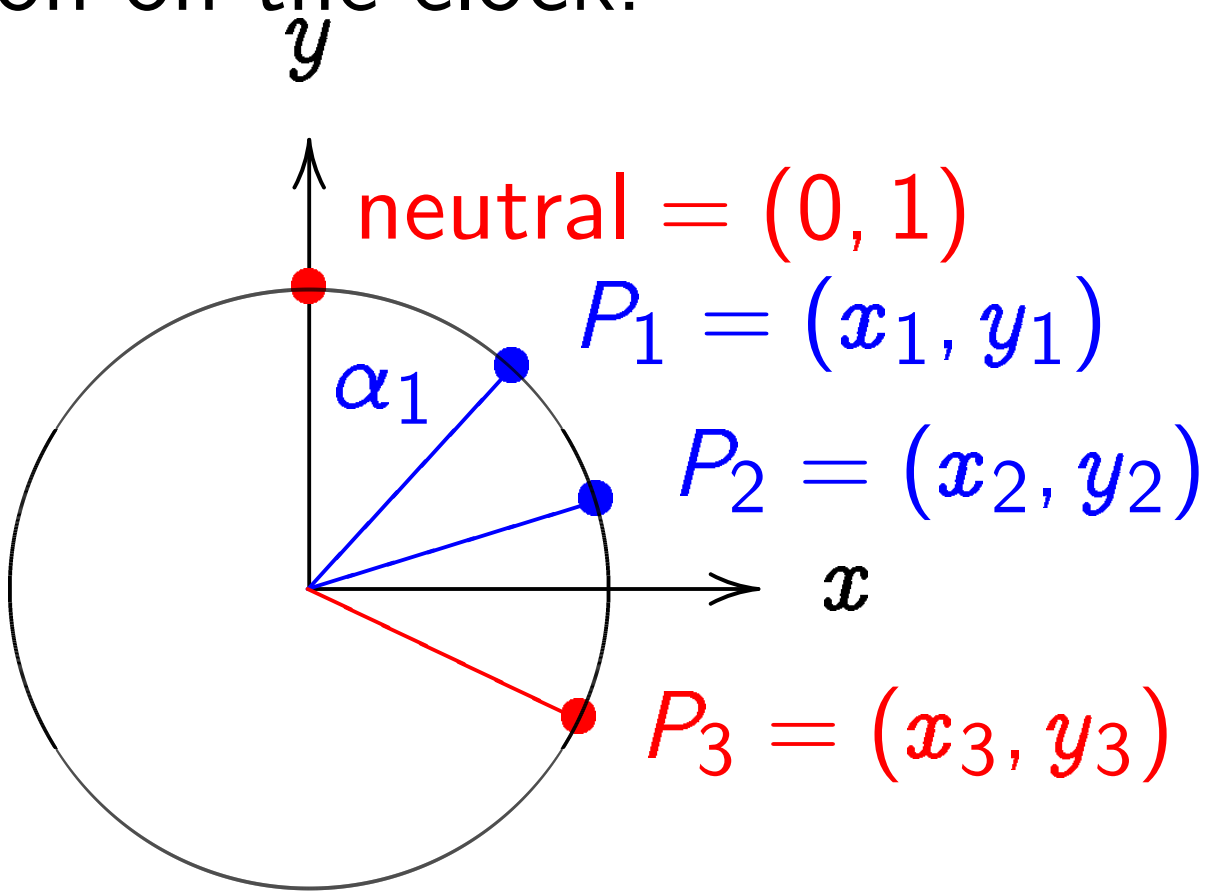

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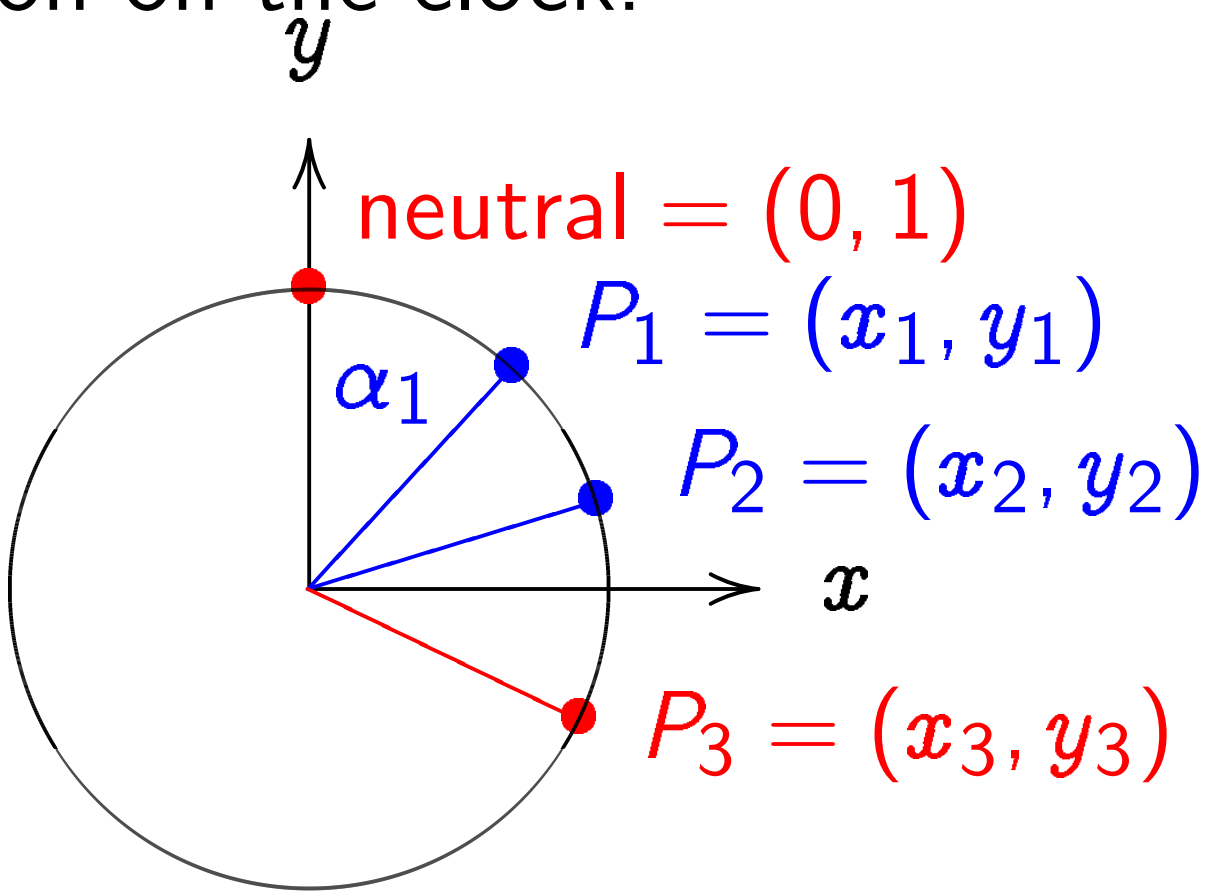

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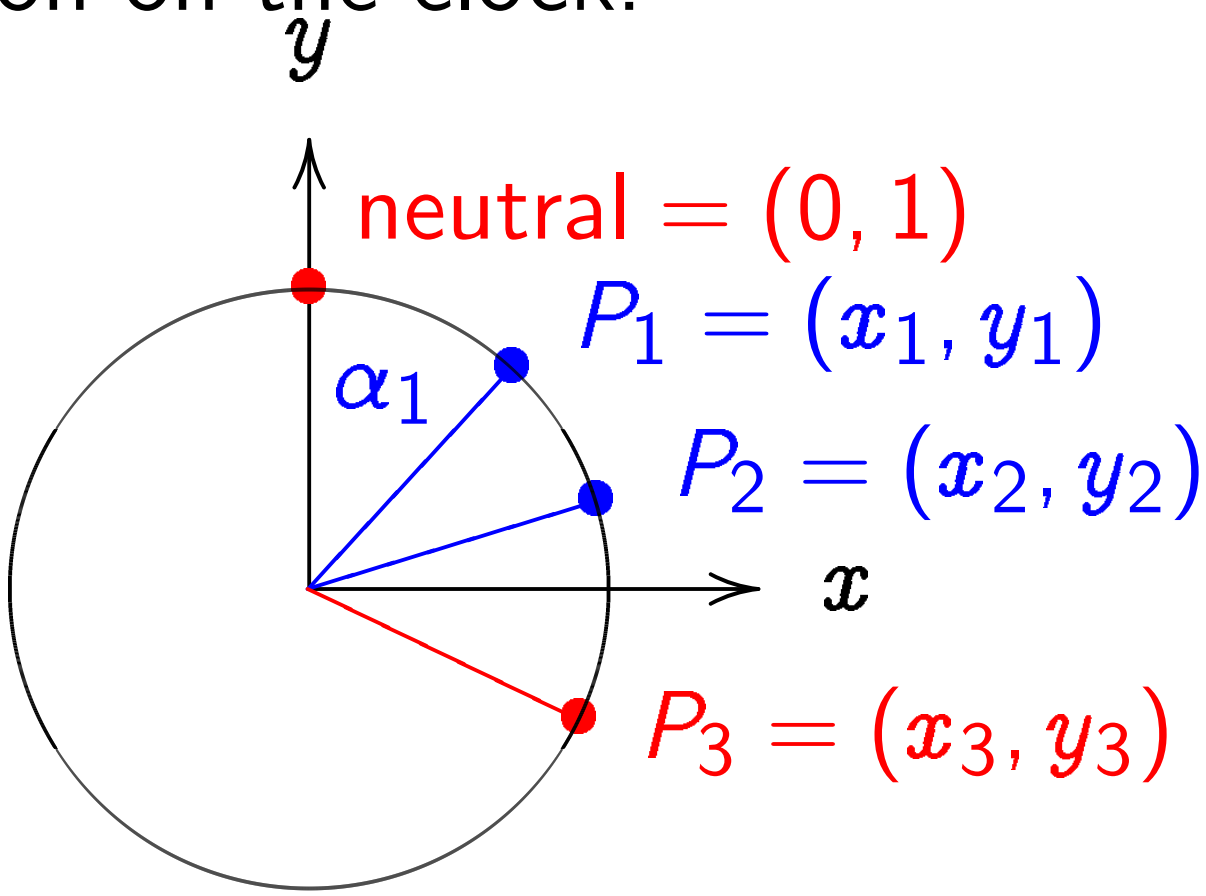

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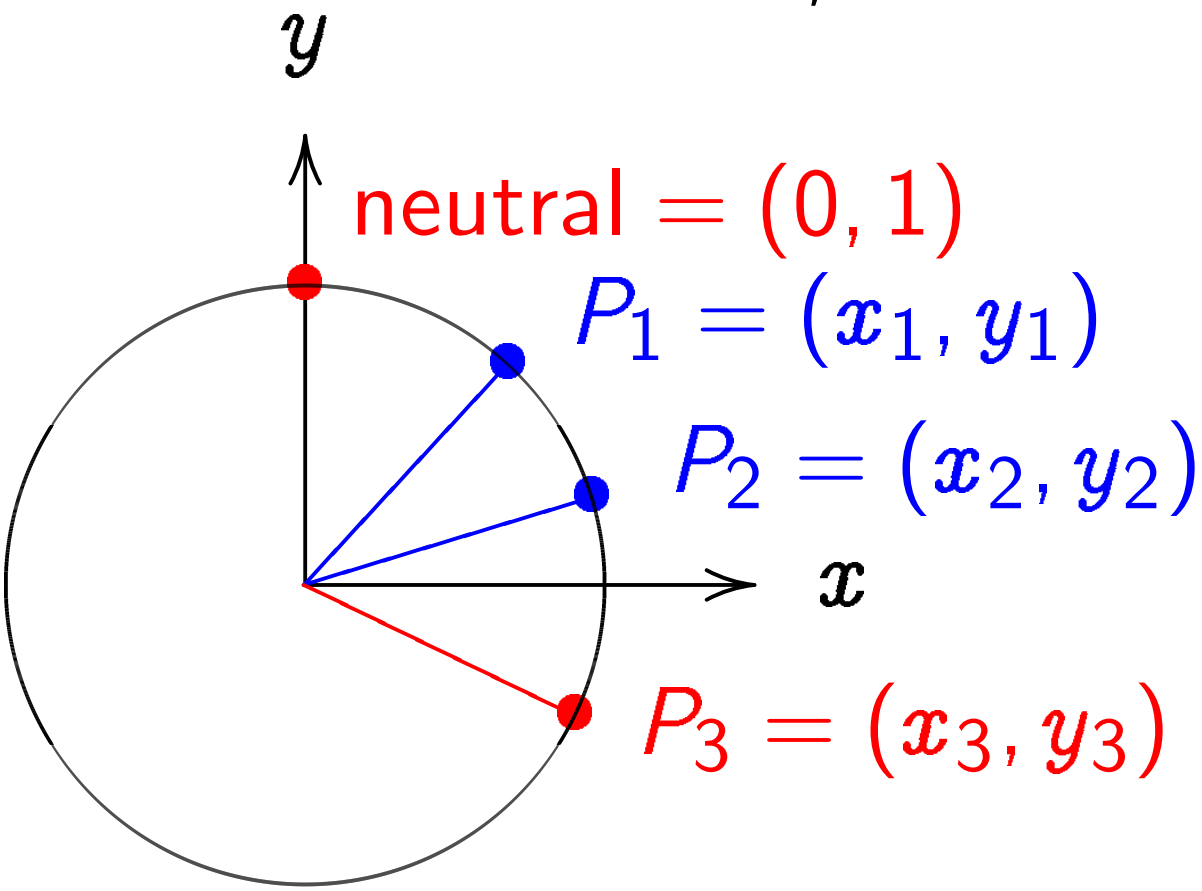
$$(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2,$$
$$\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2).$$

Adding two points corresponds
to adding the angles α_1 and α_2.
Angles modulo 360° are a group,
so points on clock are a group.

Neutral element: angle $\alpha = 0$;
point $(0, 1)$; "12:00".
The point with $\alpha = 180^\circ$
has order 2 and equals 6:00.
3:00 and 9:00 have order 4.
Inverse of point with α
is point with $-\alpha$
since $\alpha + (-\alpha) = 0$.
There are many more points
where angle α is not "nice."

Clock addition without sin, cos:



Use Cartesian coordinates for addition. Addition formula for the clock $x^2 + y^2 = 1$:

sum $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$

$= (x_1y_2 + y_1x_2, y_1y_2 - x_1x_2)$.

Note $(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

$kP = \underbrace{P + P + \cdots + P}_{k \text{ copies}}$ for $k \geq 0$.

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{“7:00”}.$$

$$\text{“5:00”} + \text{“9:00”}$$
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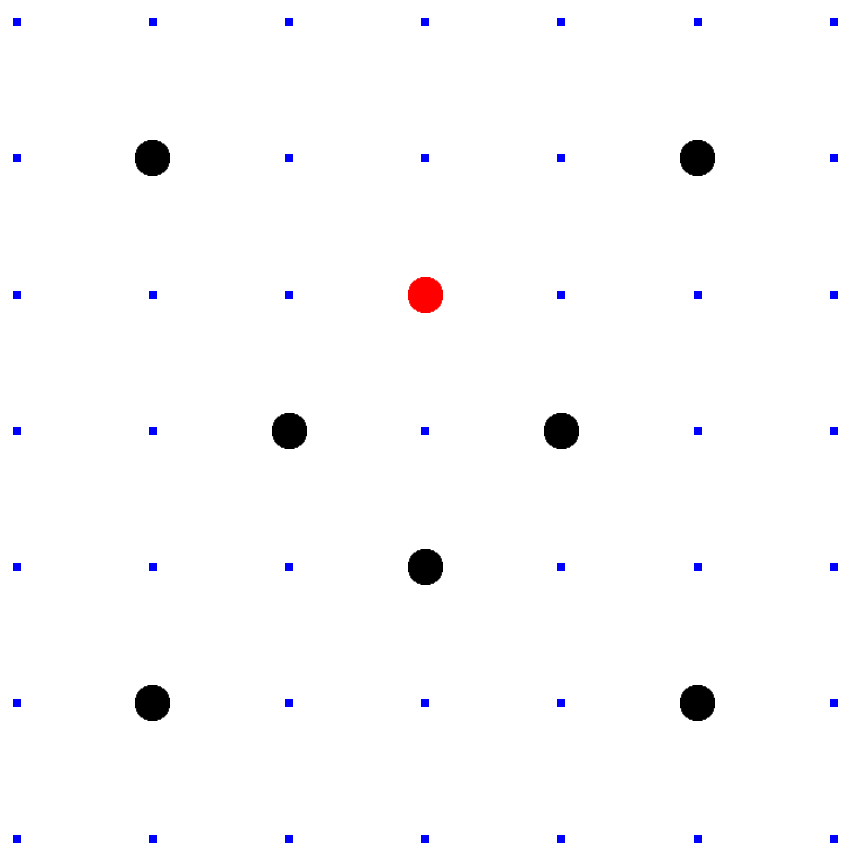
Clocks over finite fields



$\mathrm{Clock}(\mathbf{F}_7) =$

$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2 = 1\}.$

Here $\mathbf{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$= \{0, 1, 2, 3, -3, -2, -1\}$

with $+, -, \times$ modulo 7.

Larger example: Clock($\mathbf{F}_{1000003}$).

Examples of clock addition:

$2(1000, 2) = (4000, 7)$.

$4(1000, 2) = (56000, 97)$.

$8(1000, 2) = (863970, 18817)$.

$16(1000, 2) = (549438, 156853)$.

$17(1000, 2) = (951405, 877356)$.

With 30 clock additions
we computed
$n(1000, 2) = (947472, 736284)$
for some 6-digit n.
Can you figure out n?

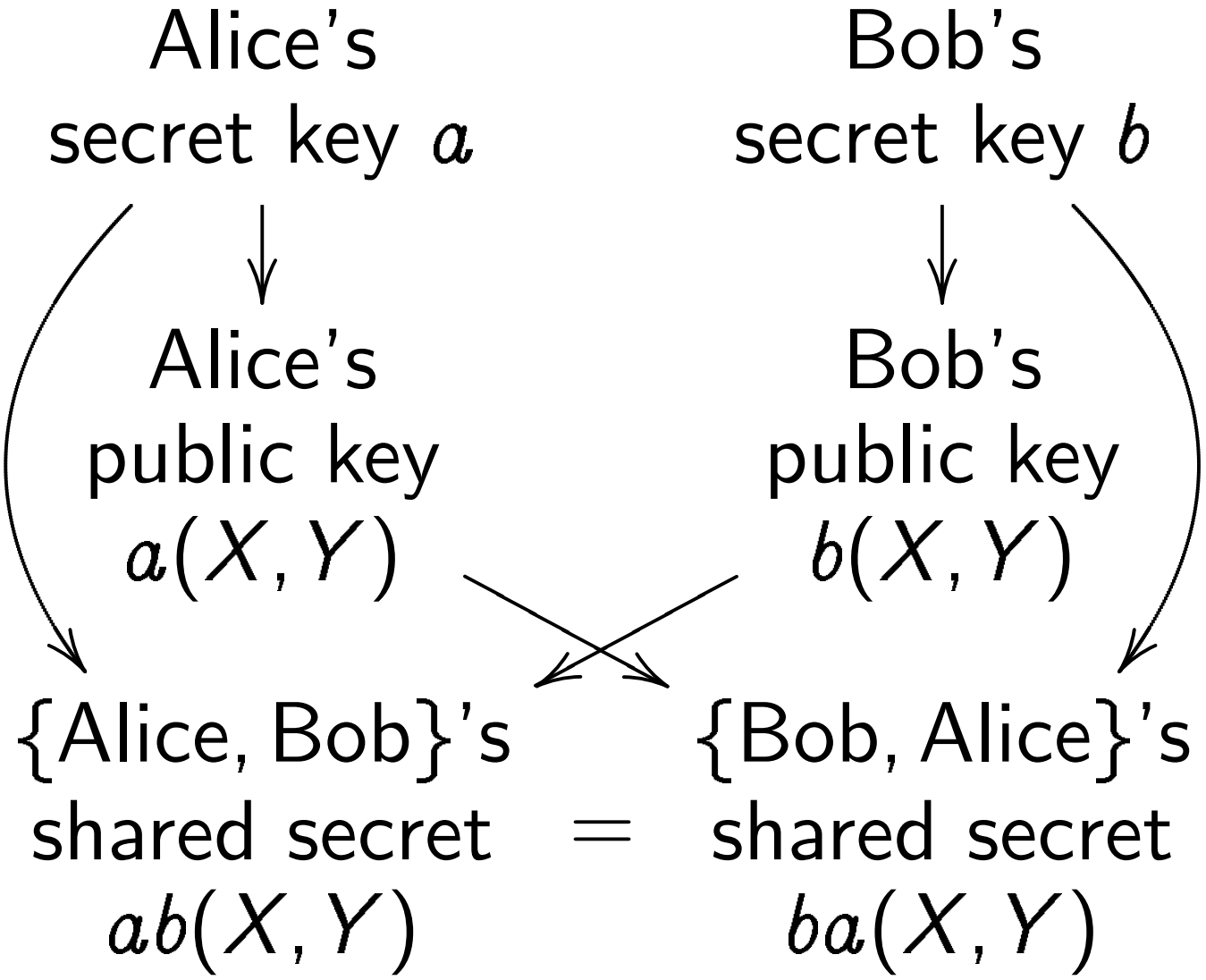
Clock cryptography

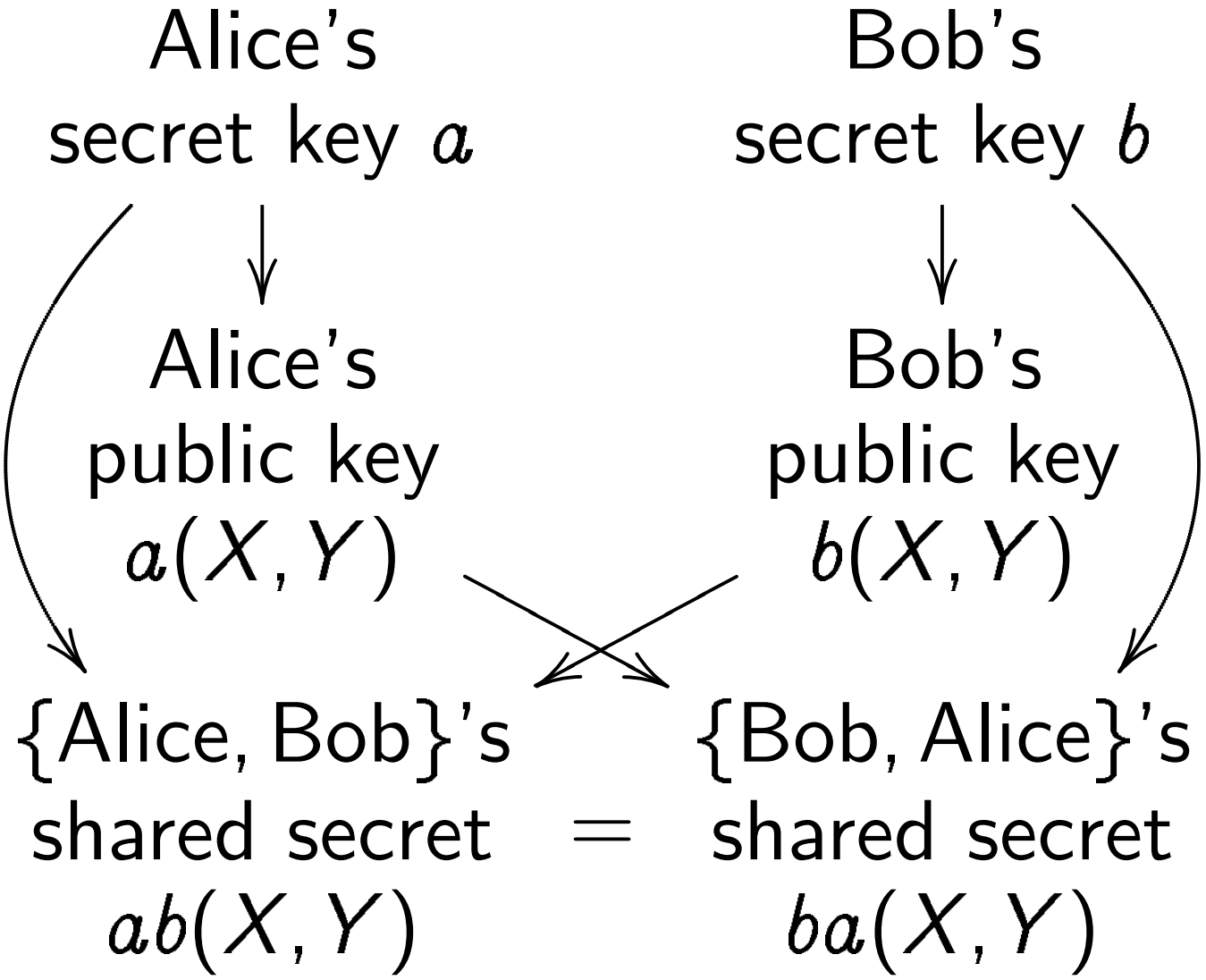
Standardize a large prime p
and some $(X, Y) \in \mathrm{Clock}(\mathbf{F}_p)$.

Alice chooses big secret a.
Computes her public key $a(X, Y)$.

Bob chooses big secret b.
Computes his public key $b(X, Y)$.

Alice computes $a(b(X, Y))$.
Bob computes $b(a(X, Y))$.
I.e., both obtain $(ab)(X, Y)$.
They use this shared value
to encrypt with AES-GCM etc.


Alice's
secret key a
Bob's
secret key b
Alice's
public key
$a(X, Y)$
Bob's
public key
$b(X, Y)$
{Alice, Bob}'s
shared secret
$ab(X, Y)$
=
{Bob, Alice}'s
shared secret
$ba(X, Y)$



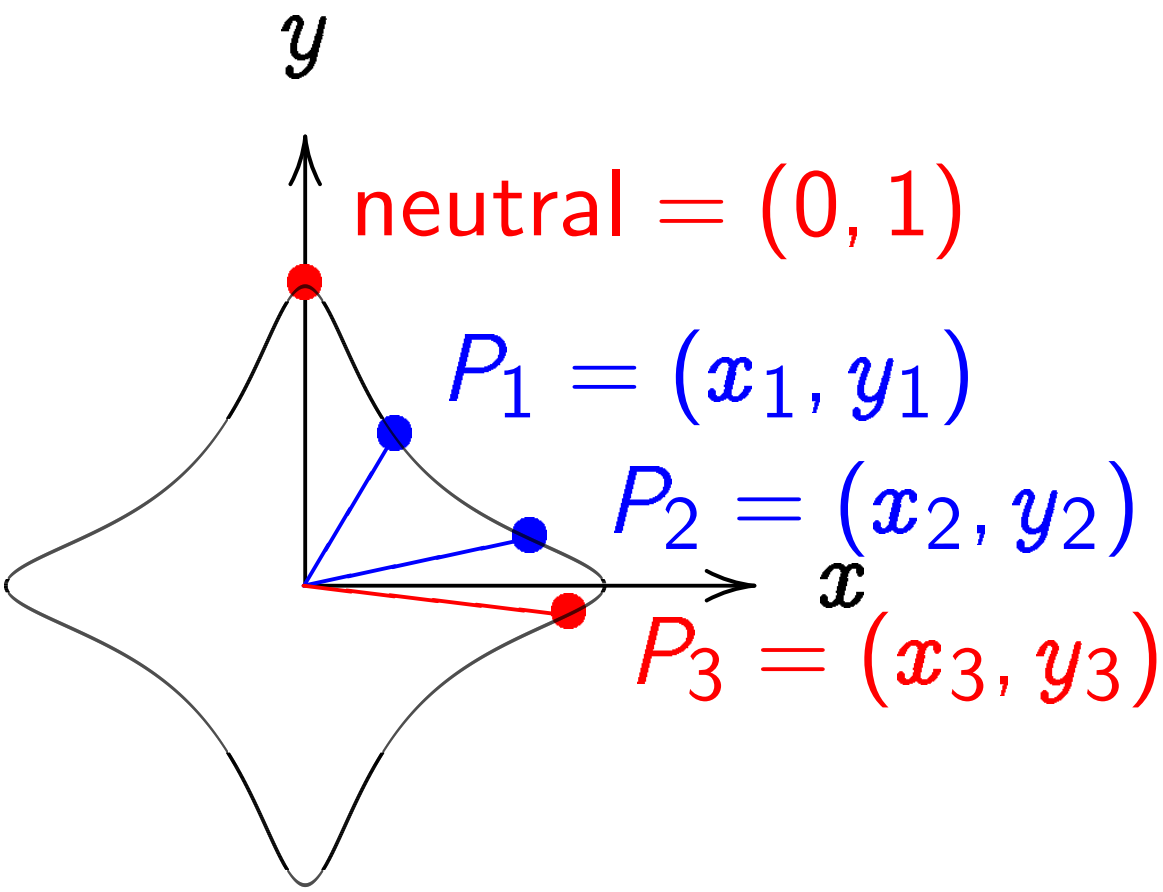
Warning: Clocks aren't elliptic!

Can attack clock cryptography, e.g., compute a from public key, by combining congruences.

To match RSA-3072 security need $p \approx 2^{1536}$.

Addition on an Edwards curve

Change the curve on which Alice and Bob work.



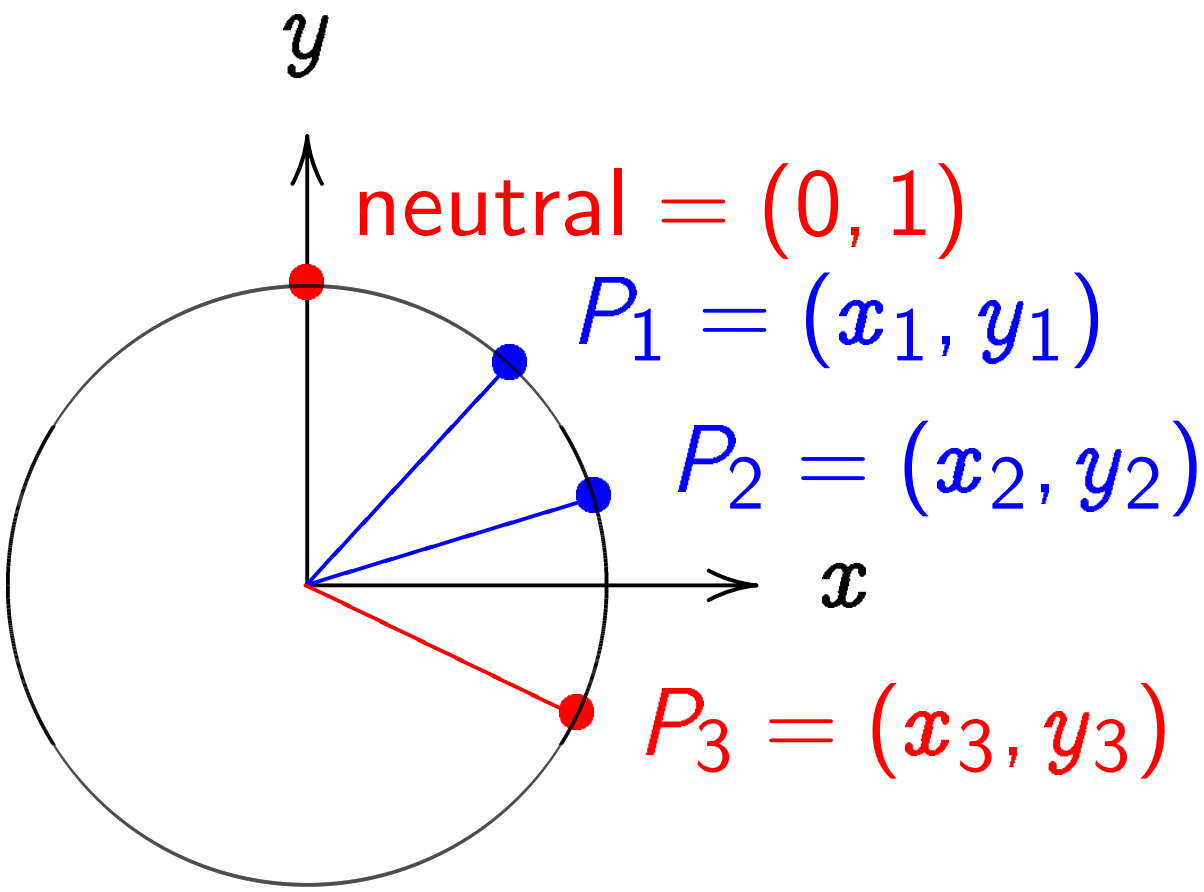
$x^2 + y^2 = 1 - 30x^2y^2$.

Sum of (x_1, y_1) and (x_2, y_2) is

$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2)$,

$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

The clock again, for comparison:



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$(x_1 y_2 + y_1 x_2,$

$y_1 y_2 - x_1 x_2)$.

"Hey, there were divisions
in the Edwards addition law!
What if the denominators are 0?"
Answer: They aren't!
If $x_i = 0$ or $y_i = 0$ then
$1 \pm 30x_1x_2y_1y_2 = 1 \neq 0$.
If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30}\ |xy| < 1$.

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then $\sqrt{30}\ |x_1y_1| < 1$
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then $\sqrt{30}\,|x_1y_1| < 1$
and $\sqrt{30}\,|x_2y_2| < 1$
so $30\,|x_1y_1x_2y_2| < 1$
so $1 \pm 30x_1x_2y_1y_2 > 0$.

The Edwards addition law
$(x_1, y_1) + (x_2, y_2) =$
$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$
is a group law for the curve
$x^2 + y^2 = 1 - 30x^2y^2$.

Some calculation required:

addition result is on curve;

addition law is associative.

Other parts of proof are easy:

addition law is commutative;

$(0, 1)$ is neutral element;

$(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

More Edwards curves

Fix an odd prime power q.

Fix a *non-square* $d \in \mathbf{F}_q$.

$$\{(x,y) \in \mathbf{F}_q \times \mathbf{F}_q : \\ x^2 + y^2 = 1 + dx^2y^2\}$$

is a commutative group with

$(x_1,y_1) + (x_2,y_2) = (x_3,y_3)$

defined by Edwards addition law:

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

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$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$

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$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$

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$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$

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$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$

$= x_1^2 + y_1^2 \pm 2x_1 y_1$

$= (x_1 \pm y_1)^2$.

Case 1: $x_2 + y_2 \neq 0$. Then

$$d = \left(\frac{x_1 \pm y_1}{x_1 y_1 (x_2 + y_2)} \right)^2,$$

contradiction.

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contradiction.

Case 2: $x_2 - y_2 \neq 0$. Then

$$d = \left(\frac{x_1 \mp y_1}{x_1 y_1 (x_2 - y_2)} \right)^2,$$

contradiction.

Case 3: $x_2 + y_2 = x_2 - y_2 = 0$. Then $x_2 = 0$ and $y_2 = 0$, contradiction.

Using ECC sensibly

Typical starting point:
Client knows secret key a
and server's public key $b(X, Y)$.
Client computes (and caches)
shared secret $ab(X, Y)$.

Client has packet for server.
Generates unique nonce.
Uses shared secret to encrypt
and authenticate packet.

Total packet overhead:
24 bytes for nonce,
16 bytes for authenticator,
32 bytes for client's public key.

Server receives packet,
sees client's public key $a(X, Y)$.
Server computes (and caches)
shared secret $ab(X, Y)$.

Server uses shared secret
to verify authenticator
and decrypt packet.

Client and server encrypt,
authenticate, verify, and decrypt
all subsequent packets
in the same way,
using the same shared secret.

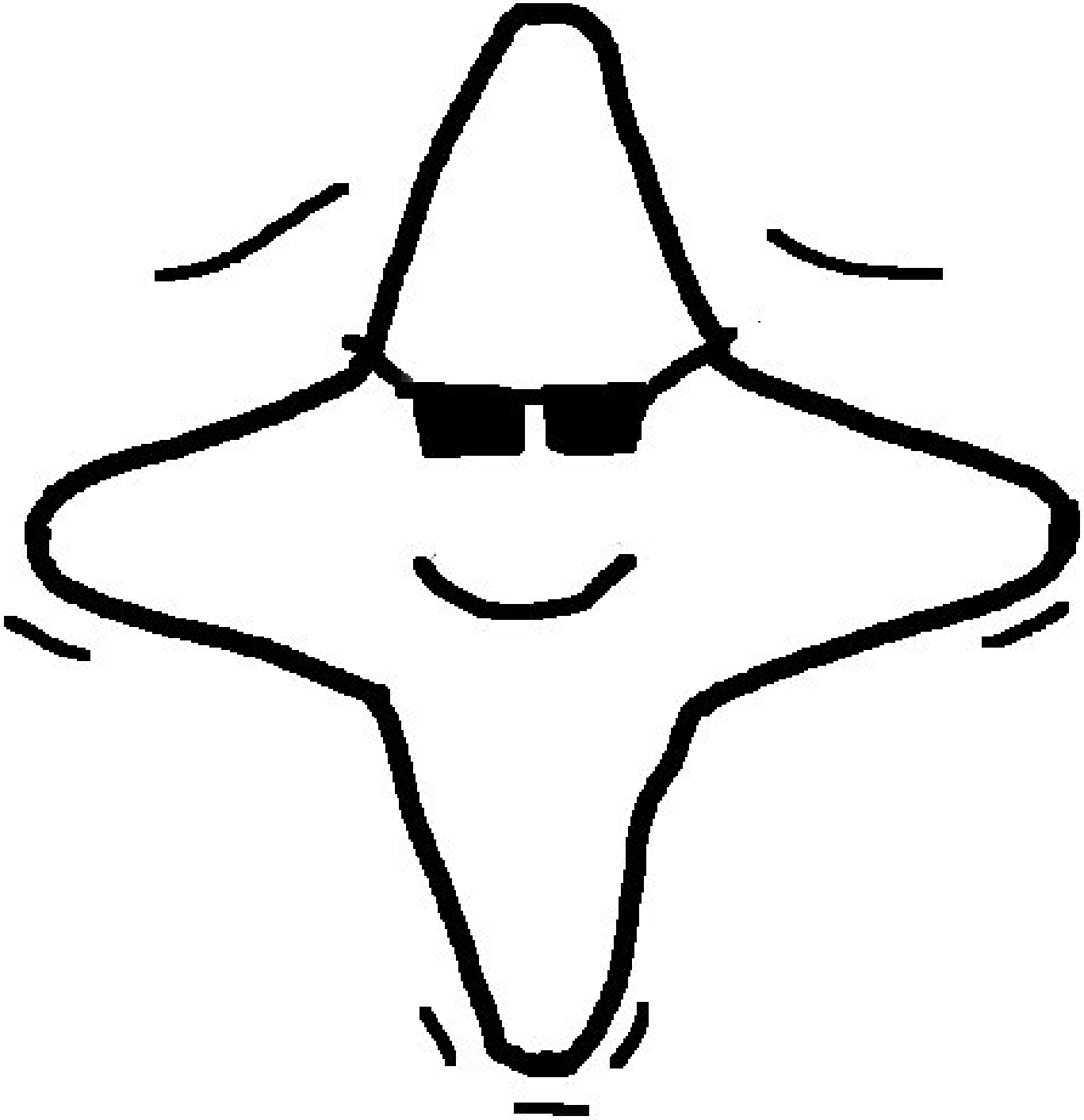
Easy-to-use packet protection:
`crypto_box` from
`nacl.cace-project.eu`.

High-security curve (Curve25519).
High-security implementation
(e.g., no secret array indices).
Extensive code validation.

Server can compute shared secrets
for 1000000 new clients
in 40 seconds of computation
on a Core 2 Quad.

Not much hope for attacker
if ECC user is running this!

Edwards curves are cool



How to compute aP

Use binary representation of a
to compute $a(X, Y)$
in $\lfloor \log_2 a \rfloor$ doublings
and at most that many additions.

E.g. $a = 23 = (10111)_2$:
$23P = 2(2(2(2P) + P) + P) + P.$
For $a = (1, a_{n-1}, \ldots, a_1, a_0)_2$;
compute *scalar multiplication*
$aP = 2(\cdots 2(2(2P + a_{n-1}P) +$
$a_{n-2}P) + \cdots + a_1P) + a_0P.$

Later: More efficient methods.

Faster group operations

Can split computation aP into additions and doublings.
Formulas $(x_1, y_1) + (x_2, y_2) =$
$((x_1y_2+y_1x_2)/(1 + dx_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1 - dx_1x_2y_1y_2))$
use expensive divisions.

Better: postpone divisions
and work with fractions.
Represent (x_1, y_1) as
$(X_1 : Y_1 : Z_1)$ with $x_1 = X_1/Z_1$
and $y_1 = Y_1/Z_1$ for $Z_1 \neq 0$.

Addition formulas in these *projective* coordinates:

$A = Z_1 \cdot Z_2;\ B = A^2;$

$C = X_1 \cdot X_2;\ D = Y_1 \cdot Y_2;$

$E = d \cdot C \cdot D;\ F = B - E;$

$G = B + E;\ X_3 = A \cdot F \cdot$

$((X_1 + Y_1) \cdot (X_2 + Y_2) - C - D);$

$Y_3 = A \cdot G \cdot (D - C);\ Z_3 = F \cdot G.$

Needs $1\mathbf{S}+10\mathbf{M}+1\mathbf{M}_d$.

Uses

$(X_1 + Y_1) \cdot (X_2 + Y_2) - C - D =$

$X_1X_2 + X_1Y_2 + Y_1X_2 + Y_1Y_2 -$

$X_1X_2 - Y_1Y_2 =$

$X_1Y_2 + Y_1X_2.$

Doubling means $P_2 = P_1$, i.e.,

$(x_1, y_1) + (x_1, y_1) =$

$((x_1y_1+y_1x_1)/(1 + dx_1x_1y_1y_1),$

$(y_1y_1-x_1x_1)/(1-dx_1x_1y_1y_1)) =$

$((2x_1y_1)/(1 + dx_1^2y_1^2),$

$(y_1^2-x_1^2)/(1 - dx_1^2y_1^2)).$

Remember $P_1 = (x_1, y_1)$

is a point on the Edwards curve,

thus $x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$ and

$2P_1 = ((2x_1y_1)/(x_1^2 + y_1^2),$

$(y_1^2-x_1^2)/(2 - (x_1^2 + y_1^2))).$

This transformation reduced

the total degree of the equation

from 4 to 2.

Doubling formulas in projective coordinates:

$$B = (X_1 + Y_1)^2;\ C = X_1^2;$$
$$D = Y_1^2;\ E = C + D;\ H = Z_1^2;$$
$$J = E - 2H;$$
$$X_3 = (B - E) \cdot J;$$
$$Y_3 = E \cdot (C - D);\ Z_3 = E \cdot J.$$

Needs 4**S**+3**M**.

Doubling formulas in projective coordinates:

$B = (X_1 + Y_1)^2;\ C = X_1^2;$

$D = Y_1^2;\ E = C + D;\ H = Z_1^2;$

$J = E - 2H;$

$X_3 = (B - E) \cdot J;$

$Y_3 = E \cdot (C - D);\ Z_3 = E \cdot J.$

Needs $4\mathbf{S}+3\mathbf{M}$.

1 doubling far less expensive than 1 addition.

Usual scalar multiplication uses many more doublings than additions.

More variations of addition:

e.g., in inverted coordinates $(X_1 : Y_1 : Z_1)$ corresponds to $x_1 = Z_1/X_1$ and $y_1 = Z_1/Y_1$.

Alternative addition formulas:
$(x_1, y_1) + (x_2, y_2) =$
$((x_1y_1 + x_2y_2)/(x_1x_2 + y_1y_2),$
$(x_1y_1 - x_2y_2)/(x_1y_2 - x_2y_1))$.

Attention: these formulas fail for doubling.

Curious fact: formulas do not involve curve parameter d.

Twisted Edwards curves

Generalization to cover more curves over given finite field $\mathbf{F}_q$:
Use $a, d \in \mathbf{F}_q^*$ with $a \neq d$
and consider twisted Edwards curve $ax^2 + y^2 = 1 + dx^2y^2$.

Why do users want more curves?
One answer: Speed!
Particularly fast choice:
$a = -1$ gives additions in $8\mathbf{M}$.

Another answer, coming later:
Smaller q for same security.
Edwards and twisted Edwards give slight bonus to attacker.

ECDSA

Users can sign messages
using Edwards curves.

Take a point P on an Edwards
curve modulo a prime $q > 2$.

ECDSA signer needs to know
the *order of* P.

There are only finitely many other
points; about q in total.
Adding P to itself will eventually
reach $(0, 1)$; let ℓ be the smallest
integer > 0 with $\ell P = (0, 1)$.
This ℓ is the order of P.

The signature scheme has as system parameters a curve E; a base point P; and a hash function h with output length at least $\lfloor \log_2 \ell \rfloor + 1$.
Alice's secret key is an integer a and her public key is $P_A = aP$.

To sign message m,
Alice computes $h(m)$;
picks random k;
computes $R = kP = (x_1, y_1)$;
puts $r \equiv y_1 \bmod \ell$; computes $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

The signature on m is (r, s).

Anybody can verify signature
given m and (r, s):
Compute $w_1 \equiv s^{-1}h(m) \bmod \ell$
and $w_2 \equiv s^{-1} \cdot r \bmod \ell$.
Check whether the y-coordinate
of $w_1P + w_2P_A$ equals r modulo ℓ
and if so, accept signature.

Alice's signatures are valid:

$$w_1P + w_2P_A =$$
$$(s^{-1}h(m))P + (s^{-1} \cdot r)P_A =$$
$$(s^{-1}(h(m) + ra))P = kP$$

and so the y-coordinate of this
expression equals r,
the y-coordinate of kP.

Attacker's view on signatures

Anybody can produce an $R = kP$.
Alice's private key is only used in
$s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A.
Most of this course deals with methods for breaking DLPs.
Sometimes attacks are easier...

If k is known for some $m, (r, s)$ then $a \equiv (sk - h(m))/r \bmod \ell$.

If two signatures $m_1, (r, s_1)$ and $m_2, (r, s_2)$ have the same value for r: assume $k_1 = k_2$; observe $s_1 - s_2 = k_1^{-1}(h(m_1) + ra - (h(m_2) + ra))$; compute $k = (s_1 - s_2)/(h(m_1) - h(m_2))$. Continue as above.

If bits of many k's are known (biased PRNG) can attack $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$ as hidden number problem using lattice basis reduction.

Malicious signer

Alice can set up her public key so that two messages of her choice share the same signature,
i.e., she can claim to have signed m_1 or m_2 at will:
$R = (x_1, y_1)$ and $-R = (-x_1, y_1)$ have the same y-coordinate.
Thus, (r, s) fits $R = kP$,
$s \equiv k^{-1}(h(m_1) + ra) \bmod \ell$ and
$-R = (-k)P$,
$s \equiv -k^{-1}(h(m_2) + ra) \bmod \ell$ if
$a \equiv -(h(m_1)+h(m_2))/2r \bmod \ell$.

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(Easy tweak: include bit of x_1.)

Elliptic curves

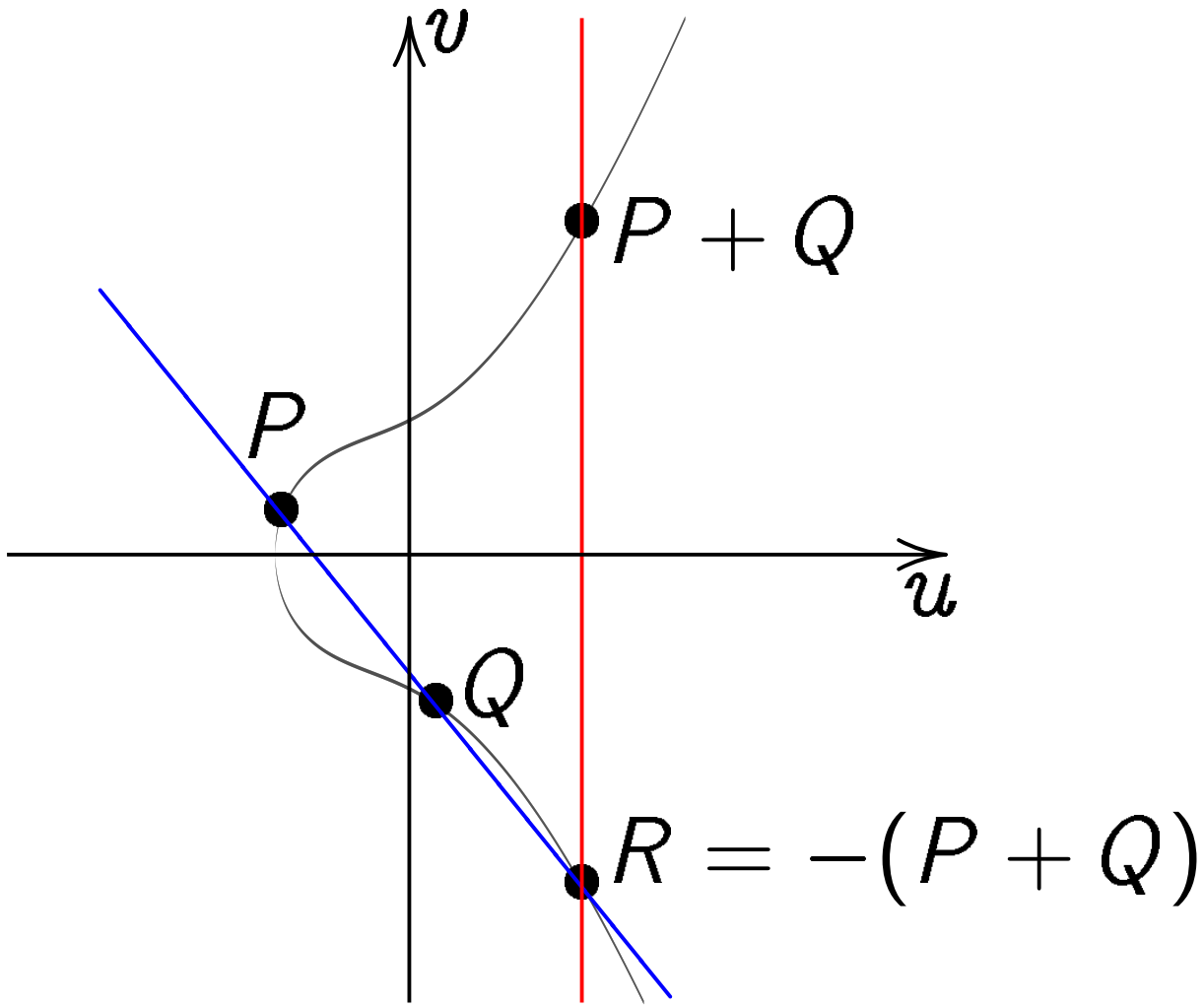
Why do we talk about
Edwards curves in this course?
Edwards curves are elliptic;
easiest way to understand
elliptic curves is Edwards.

But more elliptic curves exist!
Most common representation:
Weierstrass curve
$v^2 = u^3 + a_2u^2 + a_4u + a_6$.

(Weierstrass has different
meaning in characteristic 2
but for now we use odd char.)

Addition on Weierstrass curve

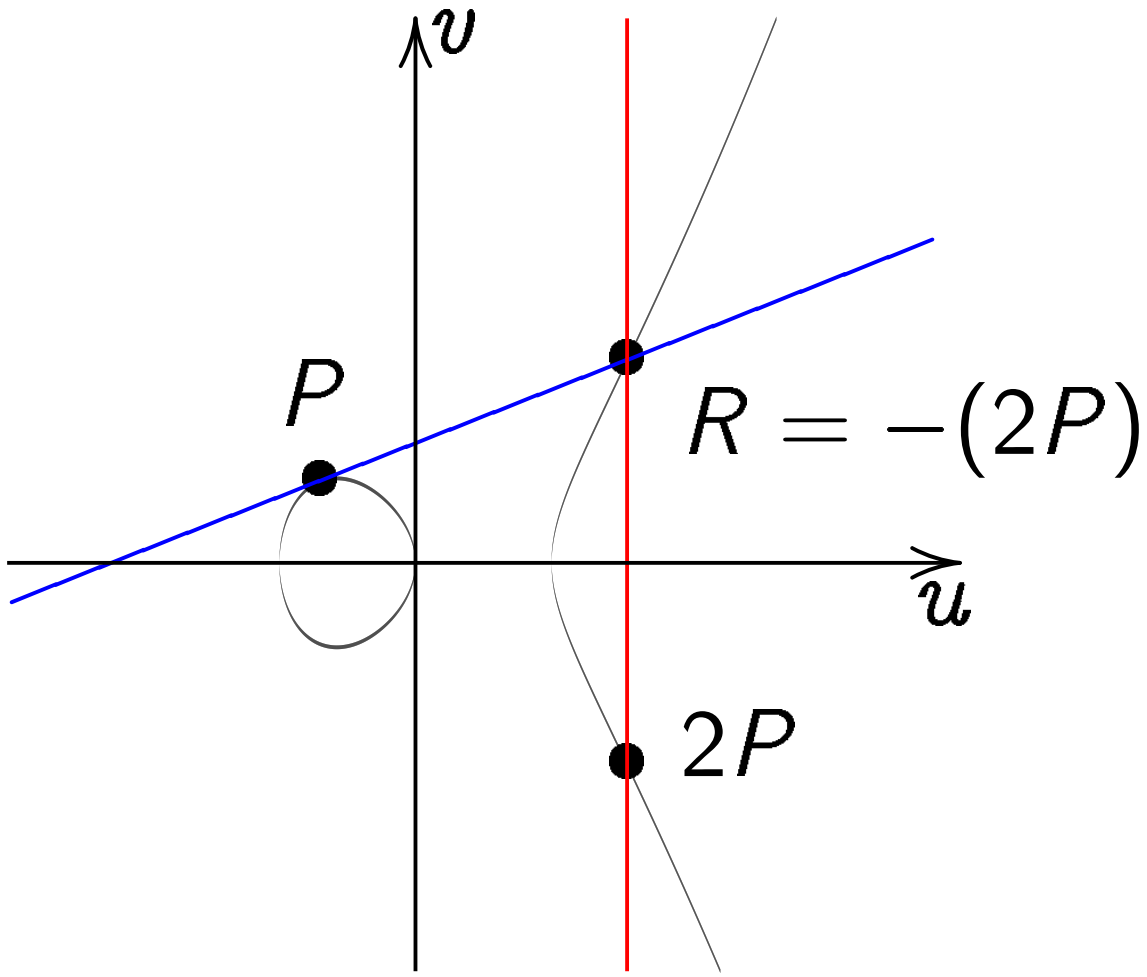
$$v^2 = u^3 + u^2 + u + 1$$



Slope $\lambda = (v_2 - v_1)/(u_2 - u_1)$
(note $u_1 \neq u_2$).

Doubling on Weierstrass curve

$v^2 = u^3 - u$



Slope $\lambda = (3u_1^2 - 1)/(2v_1)$.

General addition:
$(u_P, v_P) + (u_R, v_R) =$
$(u_{P+R}, v_{P+R}) =$
$(\lambda^2 - u_P - u_R, \lambda(u_P - u_{P+R}) - v_P)$.

$u_P \neq u_R$, "addition":
$\lambda = (v_R - v_P)/(u_R - u_P)$.
Total cost $1\mathbf{I} + 2\mathbf{M} + 1\mathbf{S}$.

$P = R$ and $v_P \neq 0$, "doubling":
$\lambda = (3u_P^2 + 2a_2u_P + a_4)/(2v_P)$.
Total cost $1\mathbf{I} + 2\mathbf{M} + 2\mathbf{S}$.

Also handle some exceptions:
$(u_P, v_P) = (u_R, -v_R)$;
inputs at ∞.

Birational equivalence

Starting from point (x, y)
on $x^2 + y^2 = 1 + dx^2y^2$:

Define $A = 2(1 + d)/(1 - d)$,
$B = 4/(1 - d)$;
$u = (1 + y)/(B(1 - y))$,
$v = u/x = (1 + y)/(Bx(1 - y))$.
(Skip a few exceptional points.)

Then (u, v) is a point on
a Weierstrass curve:
$v^2 = u^3 + (A/B)u^2 + (1/B^2)u$.

Easily invert this map:
$x = u/v$, $y = (Bu - 1)/(Bu + 1)$.

Attacker can transform Edwards curve to Weierstrass curve and vice versa; $n(x, y) \mapsto n(u, v)$.
$\Rightarrow$ Same discrete-log security!
Can choose curve representation so that implementation of attack is faster/easier.

System designer can choose curve representation so that protocol runs fastest; no need to worry about security degradation.

Optimization targets are different.
For now: understand designer's choices. After that: attacks!

Faster group operations

Designer has choice of
curve representation and
point representation.

Most protocols do
a full scalar multiplication,
many doublings and additions,
before they need a
unique representative.

Can double and add using
inversion-free systems such as
projective Edwards coordinates.
Faster, but more storage.

There are many perspectives on elliptic-curve computations.
Early development:

1984 (published 1987) Lenstra:
ECM, the elliptic-curve method
of factoring integers.

1984 (published 1985) Miller,
and independently
1984 (published 1987) Koblitz:
Elliptic-curve cryptography.

Bosma, Goldwasser–Kilian,
Chudnovsky–Chudnovsky, Atkin:
elliptic-curve primality proving.

The Edwards perspective is new!

1761 Euler, 1866 Gauss
introduced an addition law
for $x^2 + y^2 = 1 - x^2y^2$,
the "lemniscatic elliptic curve."

2007 Edwards generalized to
many curves $x^2+y^2 = 1+c^4x^2y^2$.
Theorem: have now obtained
all elliptic curves over $\overline{\mathbf{Q}}$.

2007 Bernstein–Lange:
Edwards addition law is complete
for $x^2 + y^2 = 1 + dx^2y^2$ if $d \neq$ ■;
and gives new ECC speed records.

Representing curve points

Crypto 1985, Miller, "Use of elliptic curves in cryptography":

Given $n \in \mathbf{Z}$, $P \in E(\mathbf{F}_q)$, division-polynomial recurrence computes $nP \in E(\mathbf{F}_q)$
"in $26 \log_2 n$ multiplications";
but can do better!

"It appears to be best to represent the points on the curve in the following form:
Each point is represented by the triple (x, y, z) which corresponds to the point $(x/z^2, y/z^3)$."

1986 Chudnovsky–Chudnovsky,
"Sequences of numbers
generated by addition
in formal groups
and new primality
and factorization tests":

"The crucial problem becomes
the choice of the model
of an algebraic group variety,
where computations mod p
are the least time consuming."

Most important computations:
ADD is $P, Q \mapsto P + Q$.
DBL is $P \mapsto 2P$.

"It is preferable to use
models of elliptic curves
lying in low-dimensional spaces,
for otherwise the number of
coordinates and operations is
increasing. This limits us ... to
4 basic models of elliptic curves."

Short Weierstrass:
$y^2 = x^3 + ax + b$.

Jacobi intersection:
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

Jacobi quartic: $y^2 = x^4 + 2ax^2 + 1$.

Hessian: $x^3 + y^3 + 1 = 3dxy$.

Optimizing Jacobian coordinates

For "traditional" $(X/Z^2, Y/Z^3)$
on $y^2 = x^3 + ax + b$:
1986 Chudnovsky–Chudnovsky
state explicit formulas using
10**M** for DBL; 16**M** for ADD.

Consequence:
$$\approx \left(10 \lg n + 16 \frac{\lg n}{\lg \lg n}\right) \mathbf{M}$$
to compute $n, P \mapsto nP$
using sliding-windows method
of scalar multiplication.

Notation: $\lg = \log_2$.

Squaring is faster than **M**.

Here are the DBL formulas:

$$S = 4X_1 \cdot Y_1^2;$$
$$M = 3X_1^2 + aZ_1^4;$$
$$T = M^2 - 2S;$$
$$X_3 = T;$$
$$Y_3 = M \cdot (S - T) - 8Y_1^4;$$
$$Z_3 = 2Y_1 \cdot Z_1.$$

Total cost $3\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ where
S is the cost of squaring in $\mathbf{F}_q$,
D is the cost of multiplying by a.

The squarings produce
$X_1^2, Y_1^2, Y_1^4, Z_1^2, Z_1^4, M^2$.

Most ECC standards choose
curves that make formulas faster.

Curve-choice advice from
1986 Chudnovsky–Chudnovsky:

Can eliminate the 1**D**
by choosing curve with $a = 1$.

But "it is even smarter"
to choose curve with $a = -3$.

If $a = -3$ then $M = 3(X_1^2 - Z_1^4)$
$= 3(X_1 - Z_1^2) \cdot (X_1 + Z_1^2)$.
Replace 2**S** with 1**M**.

Now DBL costs 4**M** + 4**S**.

2001 Bernstein:
$3\mathbf{M} + 5\mathbf{S}$ for DBL.
$11\mathbf{M} + 5\mathbf{S}$ for ADD.

How? Easy $\mathbf{S} - \mathbf{M}$ tradeoff:
instead of computing $2Y_1 \cdot Z_1$,
compute $(Y_1 + Z_1)^2 - Y_1^2 - Z_1^2$.
DBL formulas were already
computing Y_1^2 and Z_1^2.

Same idea for the ADD formulas,
but have to scale X, Y, Z
to eliminate divisions by 2.

ADD for $y^2 = x^3 + ax + b$:
$U_1 = X_1 Z_2^2$, $U_2 = X_2 Z_1^2$,
$S_1 = Y_1 Z_2^3$, $S_2 = Y_2 Z_1^3$,
many more computations.

1986 Chudnovsky–Chudnovsky:
"We suggest to write
addition formulas involving
(X, Y, Z, Z^2, Z^3)."

Disadvantages:
Allocate space for Z^2, Z^3.
Pay $1\mathbf{S} + 1\mathbf{M}$ in ADD and in DBL.

Advantages:
Save $2\mathbf{S} + 2\mathbf{M}$ at start of ADD.
Save $1\mathbf{S}$ at start of DBL.

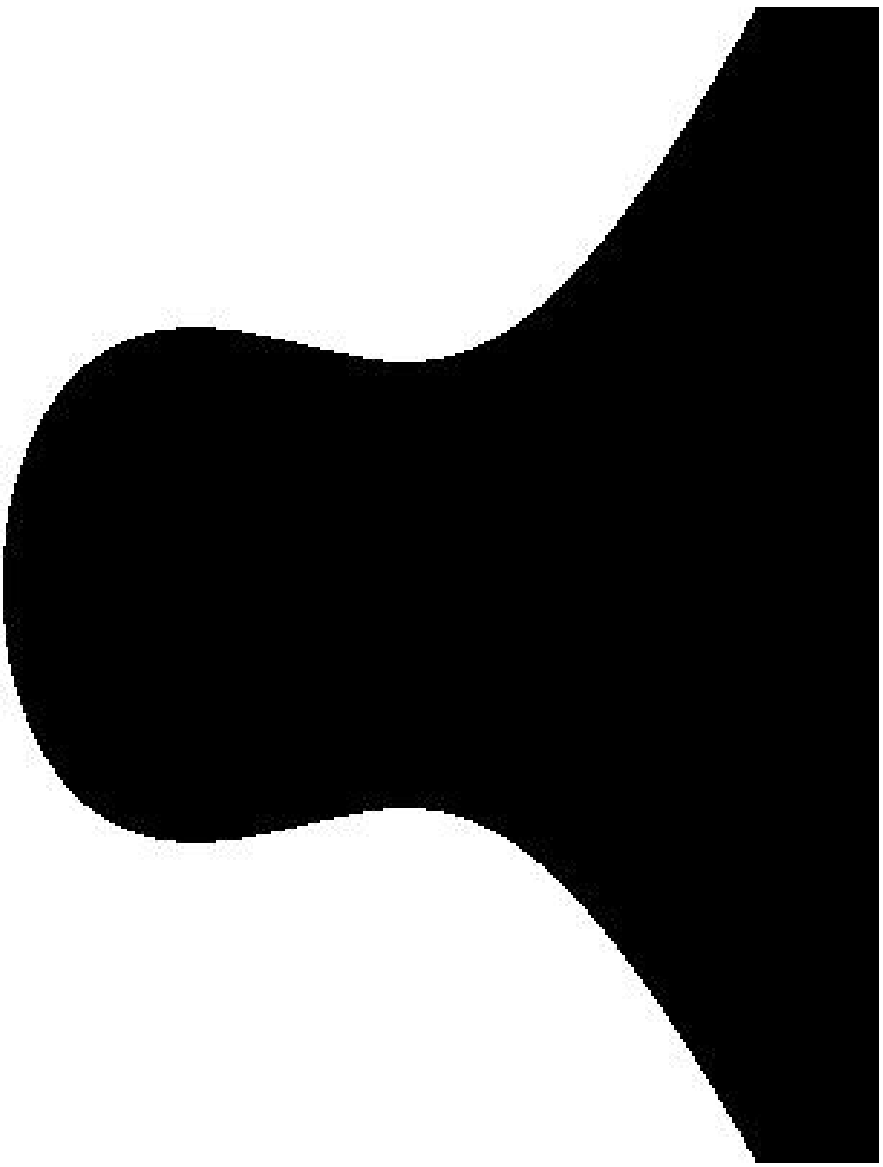
1998 Cohen–Miyaji–Ono:
Store point as $(X : Y : Z)$.
If point is input to ADD,
also cache Z^2 and Z^3.
No cost, aside from space.
If point is input to another ADD,
reuse Z^2, Z^3. Save $1\mathbf{S} + 1\mathbf{M}$!

Best Jacobian speeds today,
including $\mathbf{S} - \mathbf{M}$ tradeoffs:
$3\mathbf{M} + 5\mathbf{S}$ for DBL if $a = -3$.
$11\mathbf{M} + 5\mathbf{S}$ for ADD.
$10\mathbf{M} + 4\mathbf{S}$ for reADD.
$7\mathbf{M} + 4\mathbf{S}$ for mADD (i.e. $Z_2 = 1$).

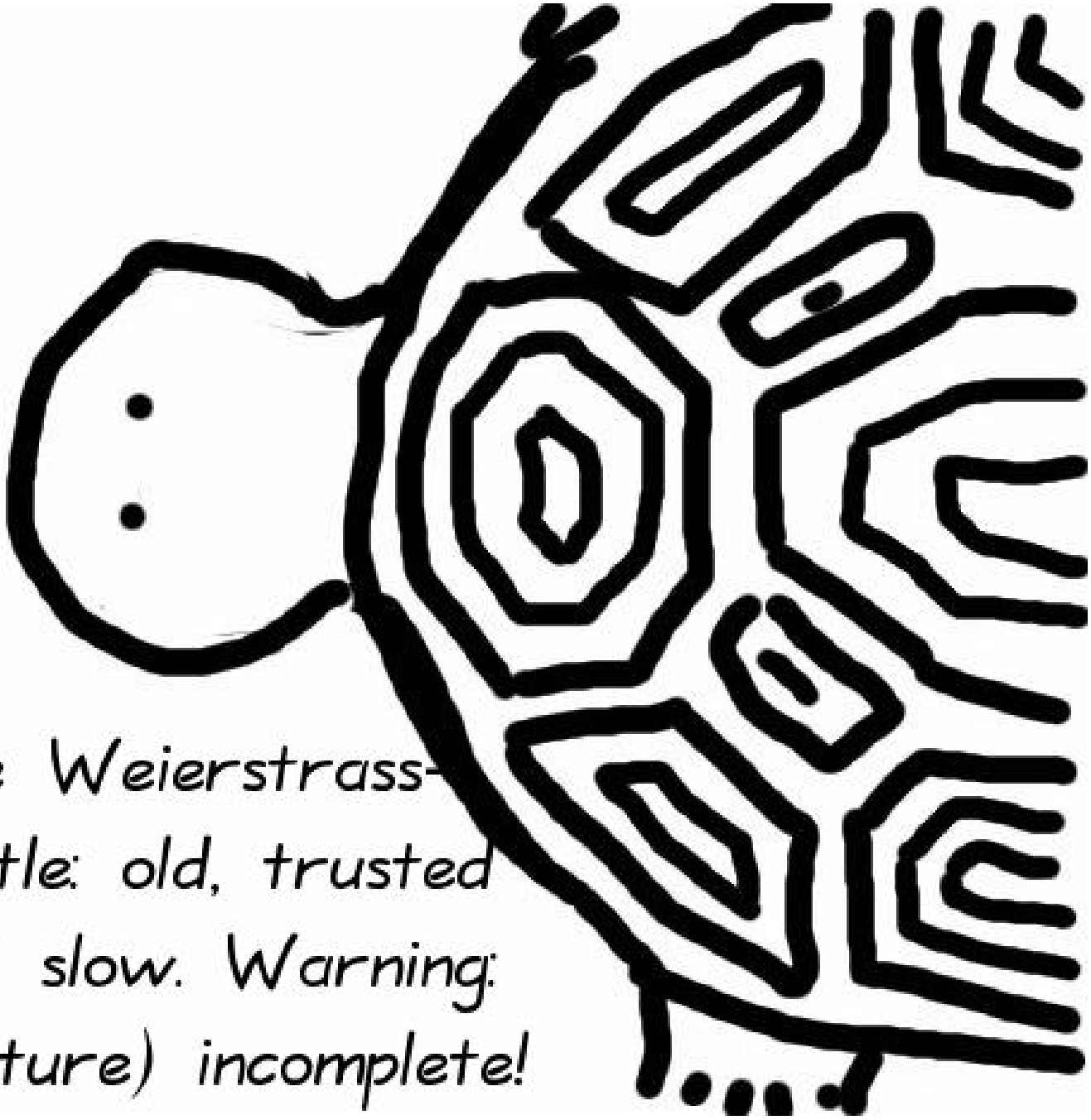
Compare to speeds for Edwards curves $x^2 + y^2 = 1 + dx^2y^2$ in projective coordinates (2007 Bernstein–Lange):
$3\mathbf{M} + 4\mathbf{S}$ for DBL.
$10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for ADD.
$9\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for mADD.

Inverted Edwards coordinates (2007 Bernstein–Lange):
$3\mathbf{M} + 4\mathbf{S} + 1\mathbf{D}$ for DBL.
$9\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for ADD.
$8\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for mADD.

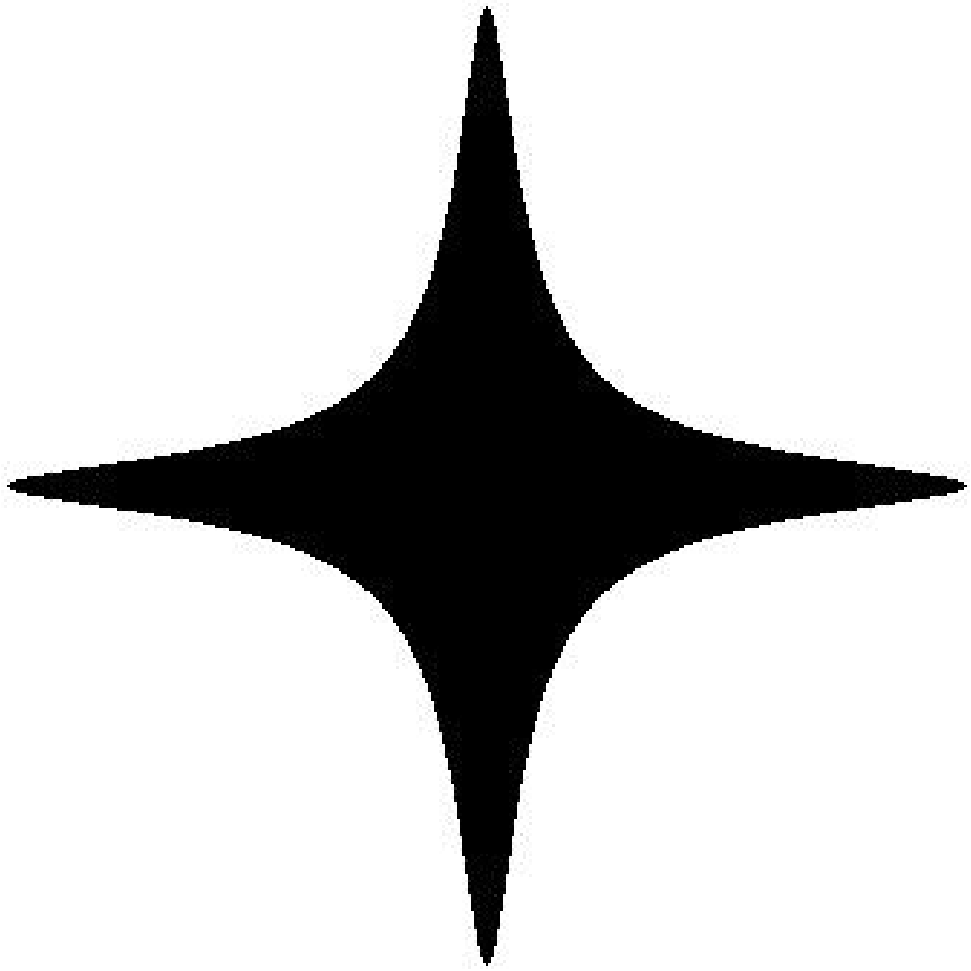
Even better speeds from extended/completed coordinates (2008 Hisil–Wong–Carter–Dawson).



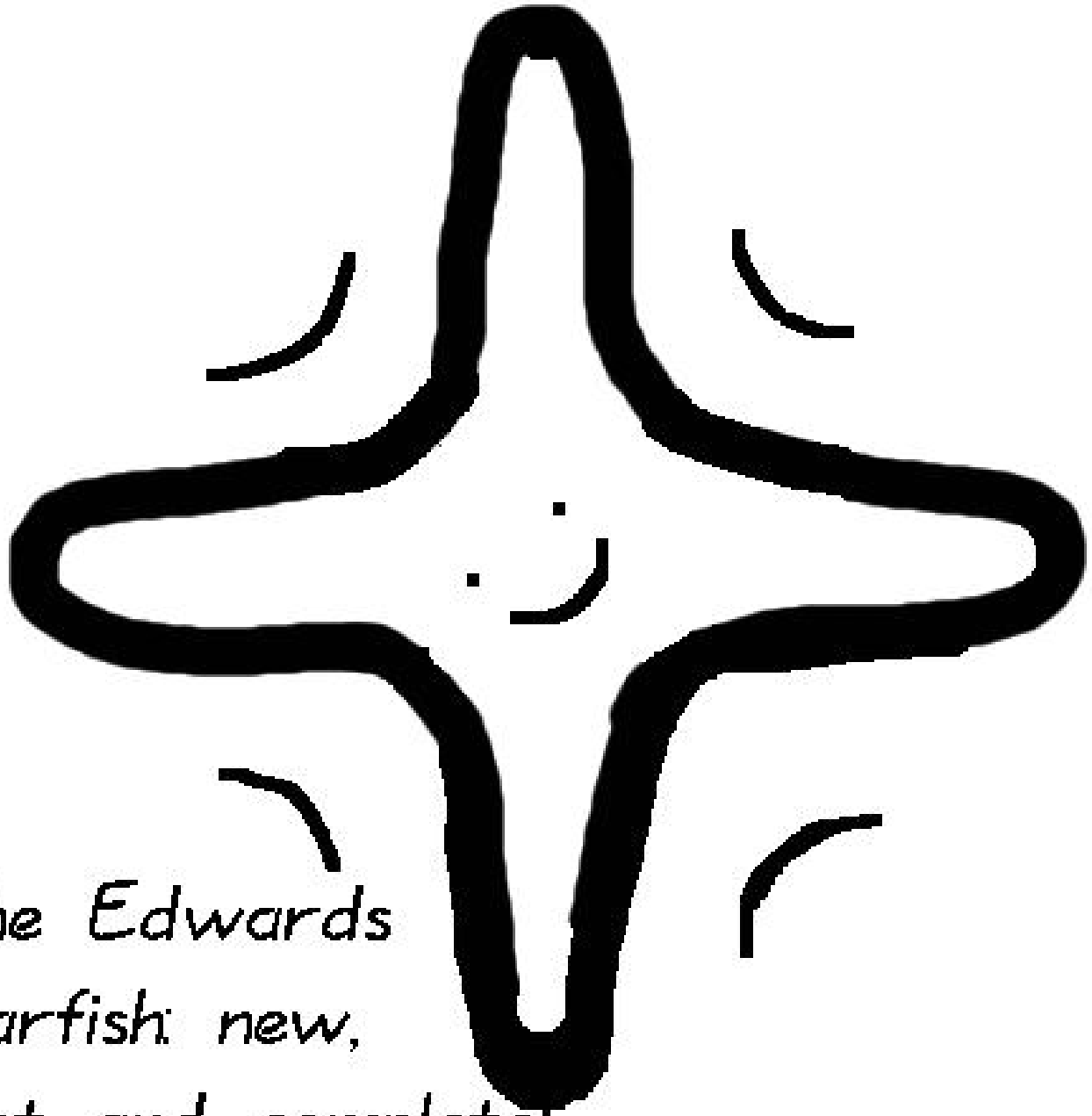
$$y^2 = x^3 - 0.4x + 0.7$$



The Weierstrass-turtle: old, trusted and slow. Warning: (picture) incomplete!



$$x^2 + y^2 = 1 - 300x^2y^2$$



The Edwards starfish: new, fast and complete!

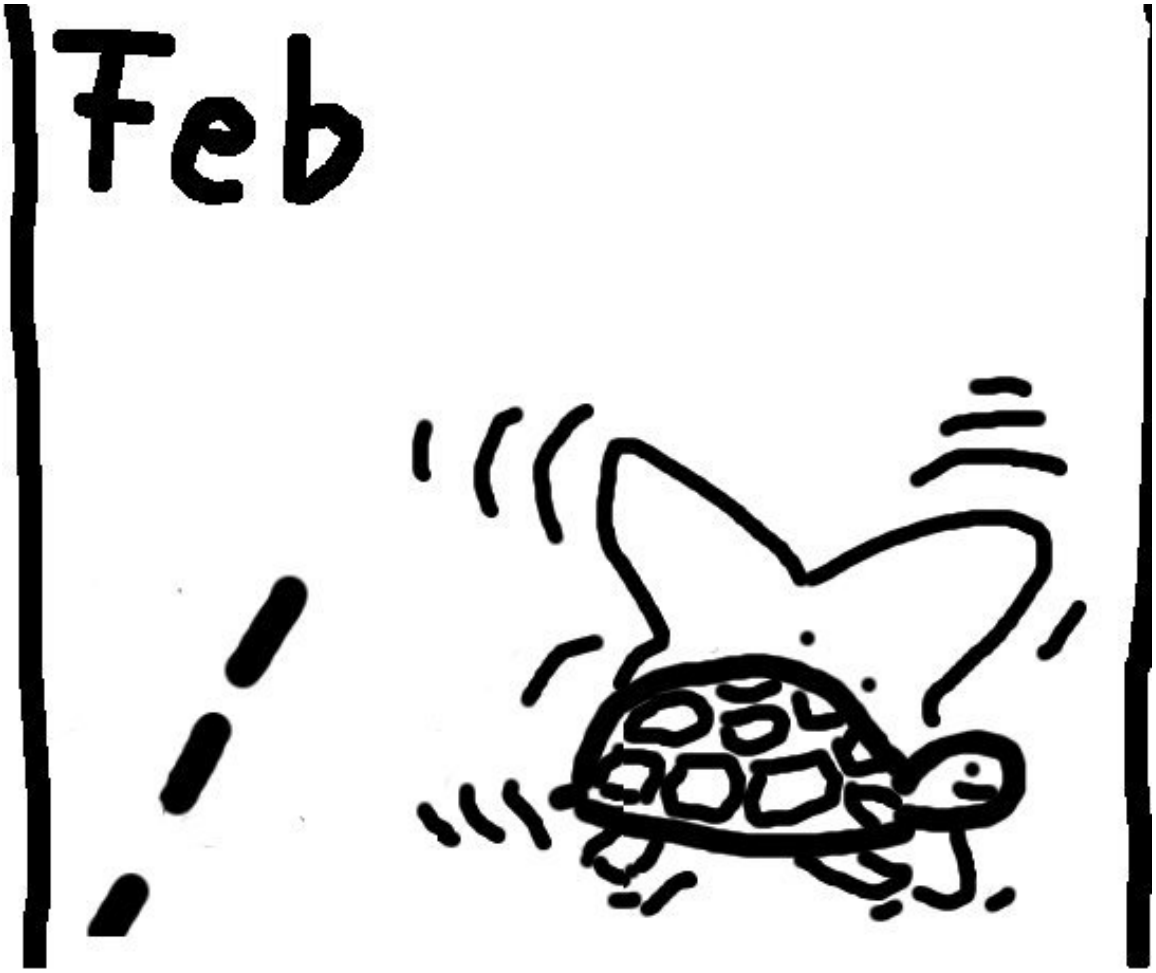
Start!

1985

Weierstrass sets off. Edwards
left behind sleeping

2007-Jan

Weierstrass has made some progress -
finally Edwards wakes up.



Exciting progress: Edwards about to overtake!!

Mar

And the winner is: Edwards!

Speed-oriented Jacobian standards

2000 IEEE "Std 1363"
uses Weierstrass curves
in Jacobian coordinates
to "provide the fastest
arithmetic on elliptic curves."
Also specifies a method of
choosing curves $y^2 = x^3 - 3x + b$.

2000 NIST "FIPS 186–2"
standardizes five such curves.

2005 NSA "Suite B" recommends
two of the NIST curves as
the only public-key cryptosystems
for U.S. government use.

Projective for Weierstrass

1986 Chudnovsky–Chudnovsky:
Speed up ADD by switching from
$(X/Z^2, Y/Z^3)$ to $(X/Z, Y/Z)$.
$7\mathbf{M} + 3\mathbf{S}$ for DBL if $a = -3$.
$12\mathbf{M} + 2\mathbf{S}$ for ADD.
$12\mathbf{M} + 2\mathbf{S}$ for reADD.

Option has been mostly ignored:
DBL dominates in ECDH etc.
But ADD dominates in
some applications: e.g.,
batch signature verification.

Montgomery curves

1987 Montgomery:

Use $by^2 = x^3 + ax^2 + x$.
Choose small $(a+2)/4$.

$$2(x_2, y_2) = (x_4, y_4)$$
$$\Rightarrow x_4 = \frac{(x_2^2 - 1)^2}{4x_2(x_2^2 + ax_2 + 1)}.$$

$$(x_3, y_3) - (x_2, y_2) = (x_1, y_1),$$
$$(x_3, y_3) + (x_2, y_2) = (x_5, y_5)$$
$$\Rightarrow x_5 = \frac{(x_2x_3 - 1)^2}{x_1(x_2 - x_3)^2}.$$

Represent (x, y)
as $(X{:}Z)$ satisfying $x = X/Z$.

$B = (X_2 + Z_2)^2$,
$C = (X_2 - Z_2)^2$,
$D = B - C$, $X_4 = B \cdot C$,
$Z_4 = D \cdot (C + D(a + 2)/4) \Rightarrow$
$2(X_2{:}Z_2) = (X_4{:}Z_4)$.

$(X_3{:}Z_3) - (X_2{:}Z_2) = (X_1{:}Z_1)$,
$E = (X_3 - Z_3) \cdot (X_2 + Z_2)$,
$F = (X_3 + Z_3) \cdot (X_2 - Z_2)$,
$X_5 = Z_1 \cdot (E + F)^2$,
$Z_5 = X_1 \cdot (E - F)^2 \Rightarrow$
$(X_3{:}Z_3) + (X_2{:}Z_2) = (X_5{:}Z_5)$.

This representation
does not allow ADD but it allows
DADD, "differential addition":
$Q, R, Q - R \mapsto Q + R$.

e.g. $2P, P, P \mapsto 3P$.
e.g. $3P, 2P, P \mapsto 5P$.
e.g. $6P, 5P, P \mapsto 11P$.

$2\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for DBL.
$4\mathbf{M} + 2\mathbf{S}$ for DADD.
Save $1\mathbf{M}$ if $Z_1 = 1$.

Easily compute $n(X_1 : Z_1)$ using
$\approx \lg n$ DBL, $\approx \lg n$ DADD.
Almost as fast as Edwards nP.
Relatively slow for $mP + nQ$ etc.

Doubling-oriented curves

2006 Doche–Icart–Kohel:

Use $y^2 = x^3 + ax^2 + 16ax$.
Choose small a.

Use $(X : Y : Z : Z^2)$
to represent $(X/Z, Y/Z^2)$.

$3\mathbf{M} + 4\mathbf{S} + 2\mathbf{D}$ for DBL.
How? Factor DBL as $\hat{\varphi}(\varphi)$
where φ is a 2-isogeny.

2007 Bernstein–Lange:
$2\mathbf{M} + 5\mathbf{S} + 2\mathbf{D}$ for DBL
on the same curves.

$12\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for ADD.
Slower ADD than other systems,
typically outweighing benefit
of the very fast DBL.

But isogenies are useful.
Example, 2005 Gaudry:
fast DBL+DADD on Jacobians of
genus-2 hyperelliptic curves,
using similar factorization.

Tricky but potentially helpful:
tripling-oriented curves
(see 2006 Doche–Icart–Kohel),
double-base chains, ...

Hessian curves

Credited to Sylvester
by 1986 Chudnovsky–Chudnovsky:

$(X : Y : Z)$ represent $(X/Z, Y/Z)$
on $x^3 + y^3 + 1 = 3dxy$.

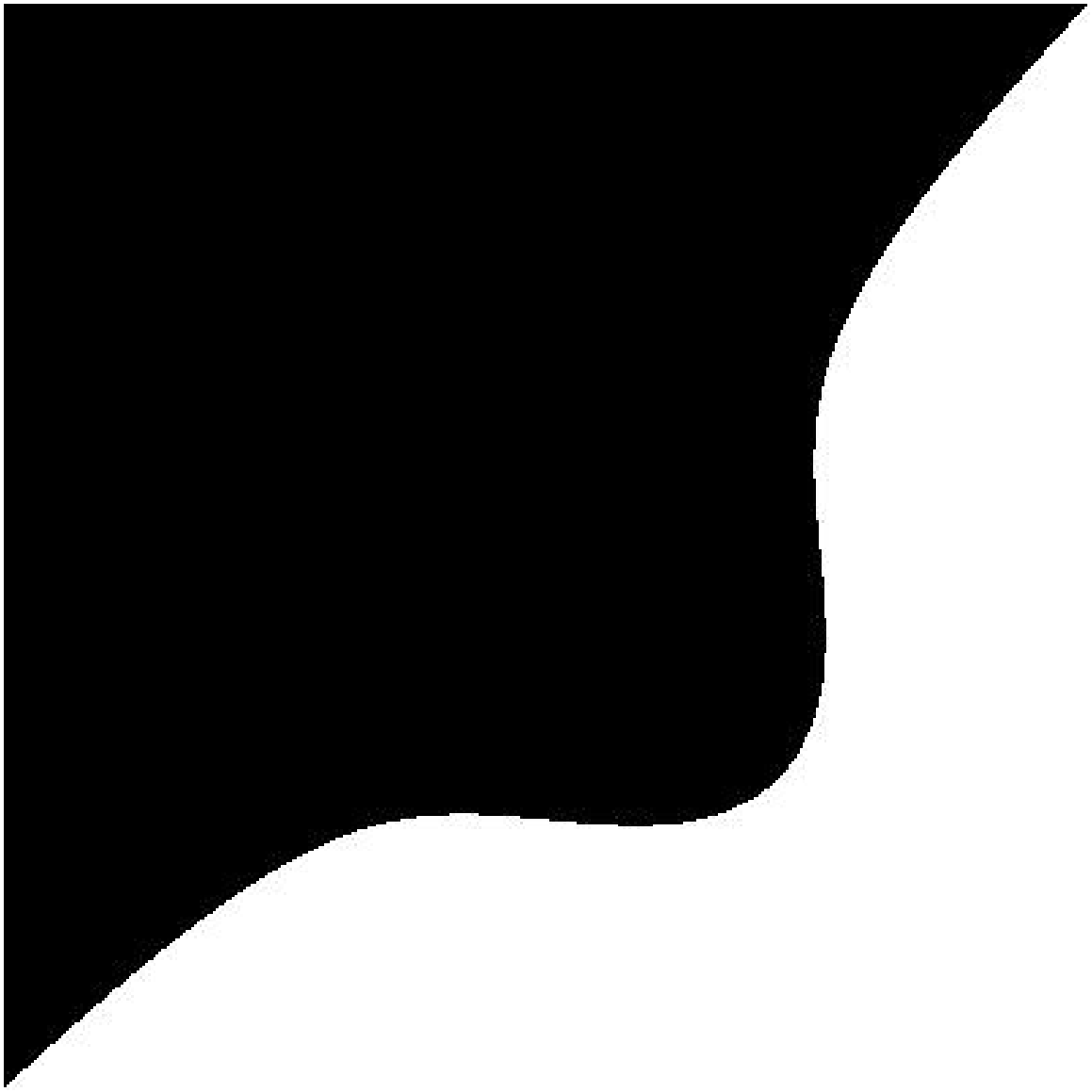
12**M** for ADD:
$X_3 = Y_1X_2 \cdot Y_1Z_2 - Z_1Y_2 \cdot X_1Y_2$,
$Y_3 = X_1Z_2 \cdot X_1Y_2 - Y_1X_2 \cdot Z_1X_2$,
$Z_3 = Z_1Y_2 \cdot Z_1X_2 - X_1Z_2 \cdot Y_1Z_2$.

6**M** + 3**S** for DBL.

2001 Joye–Quisquater:
$2(X_1 : Y_1 : Z_1) =$
$(Z_1 : X_1 : Y_1) + (Y_1 : Z_1 : X_1)$
so can use ADD to double.

"Unified addition formulas,"
helpful against side channels.
But need to permute inputs.
2009 Bernstein–Kohel–Lange:
Easily avoid permutation!

2008 Hisil–Wong–Carter–Dawson:
$(X : Y : Z : X^2 : Y^2 : Z^2$
$\quad : 2XY : 2XZ : 2YZ).$
$6\mathbf{M} + 6\mathbf{S}$ for ADD.
$3\mathbf{M} + 6\mathbf{S}$ for DBL.



$$x^3 - y^3 + 1 = 0.3xy$$


The Hessian-ray: uniform
but
not strongly so

Jacobi intersections

1986 Chudnovsky–Chudnovsky:

$(S : C : D : Z)$ represent
$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
"Tremendous advantage"
of being strongly unified.

$5\mathbf{M} + 3\mathbf{S}$ for DBL.
"Perhaps (?) ... the most
efficient duplication formulas
which do not depend on the
coefficients of an elliptic curve."

2001 Liardet–Smart:

$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.

$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:

$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008 Hisil–Wong–Carter–Dawson:

$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.

$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

Also $(S : C : D : Z : SC : DZ)$:

$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.

$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

Jacobi quartics

$(X{:}Y{:}Z)$ represent $(X/Z, Y/Z^2)$
on $y^2 = x^4 + 2ax^2 + 1$.

1986 Chudnovsky–Chudnovsky:
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.
Slow ADD.

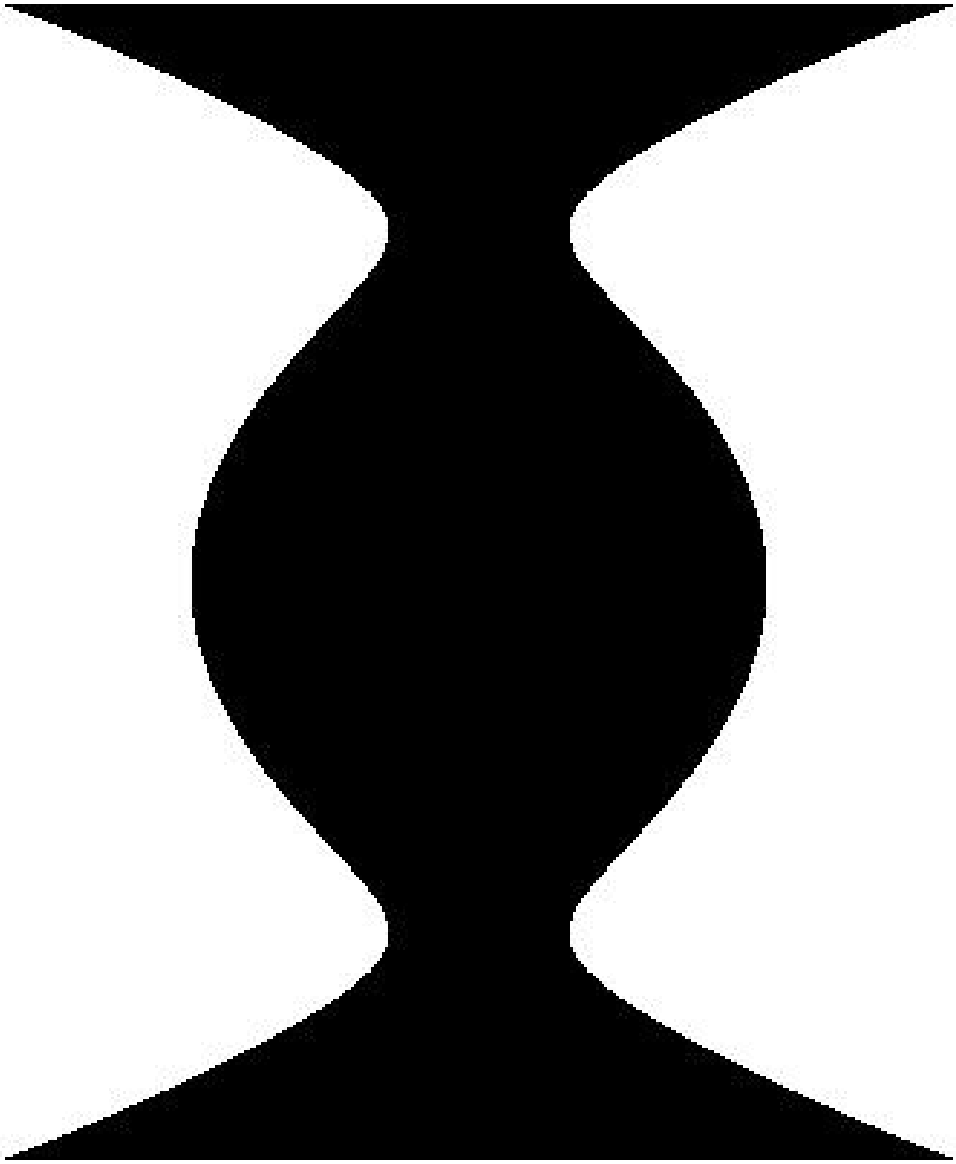
2002 Billet–Joye:
New choice of neutral element.
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD,
strongly unified.

2007 Bernstein–Lange:
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

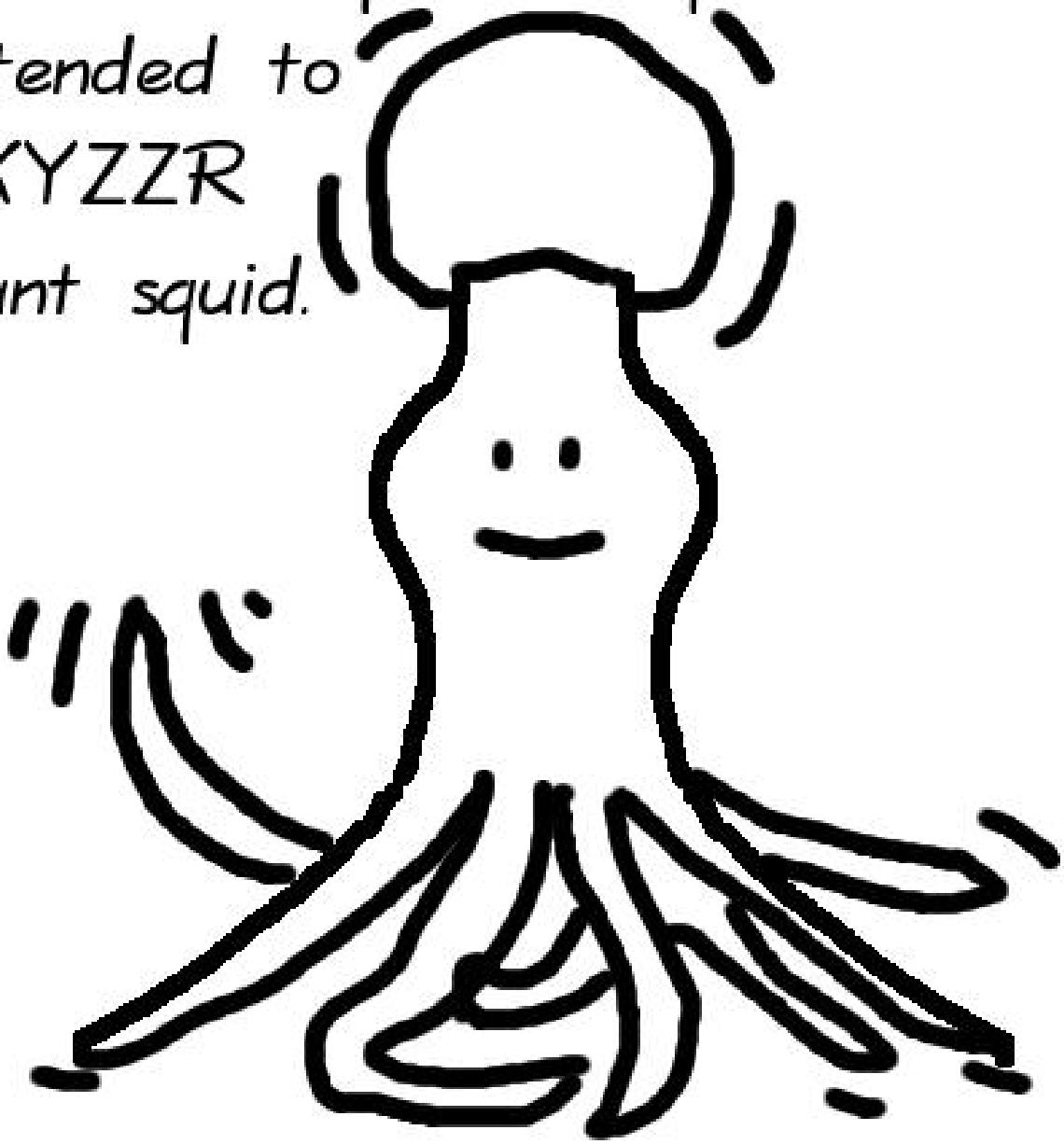
2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne,
2007 Hisil–Carter–Dawson,
2008 Hisil–Wong–Carter–Dawson:
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.
Can combine with Feng–Wu.
Competitive with Edwards!

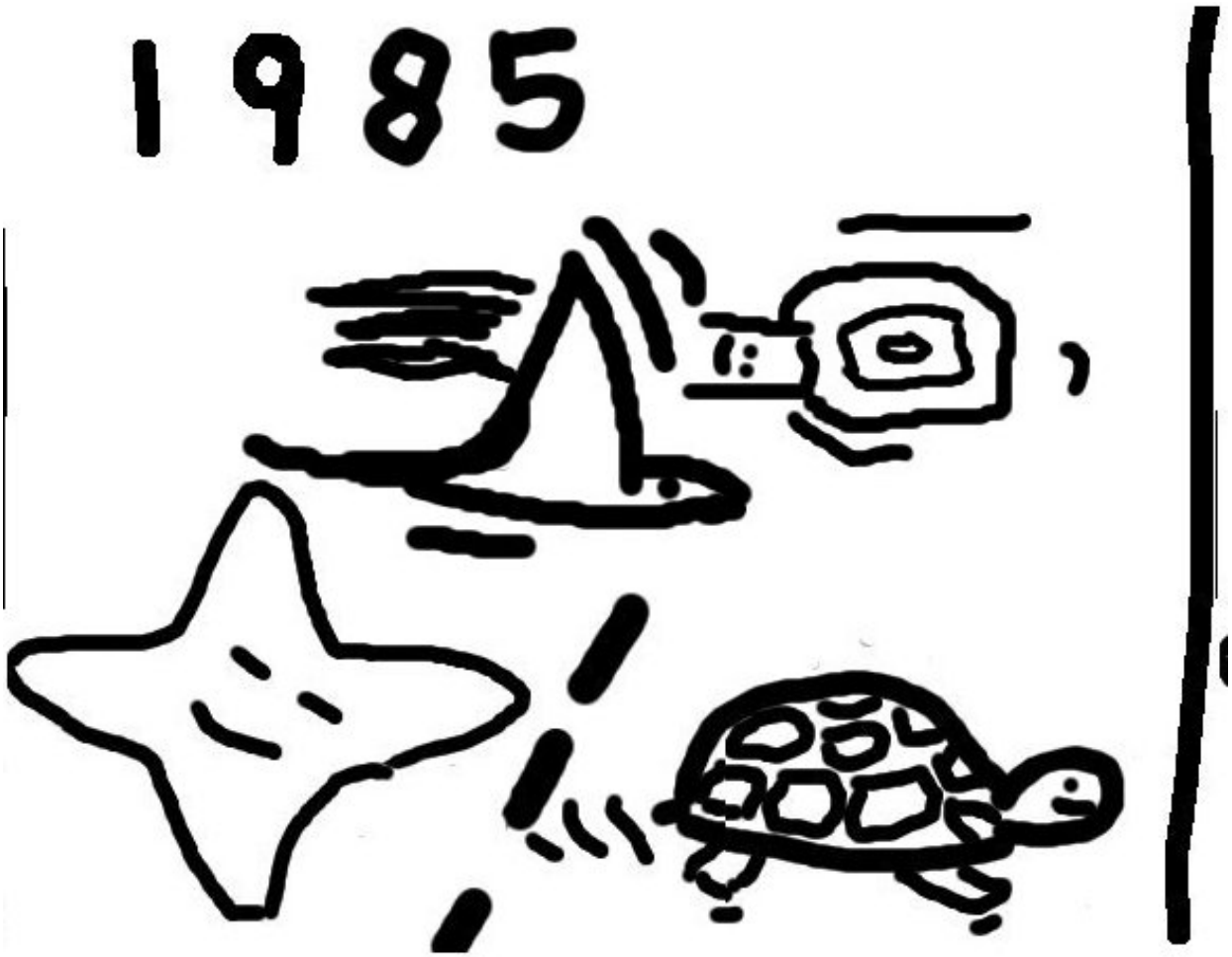


$$x^2 = y^4 - 1.9y^2 + 1$$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.

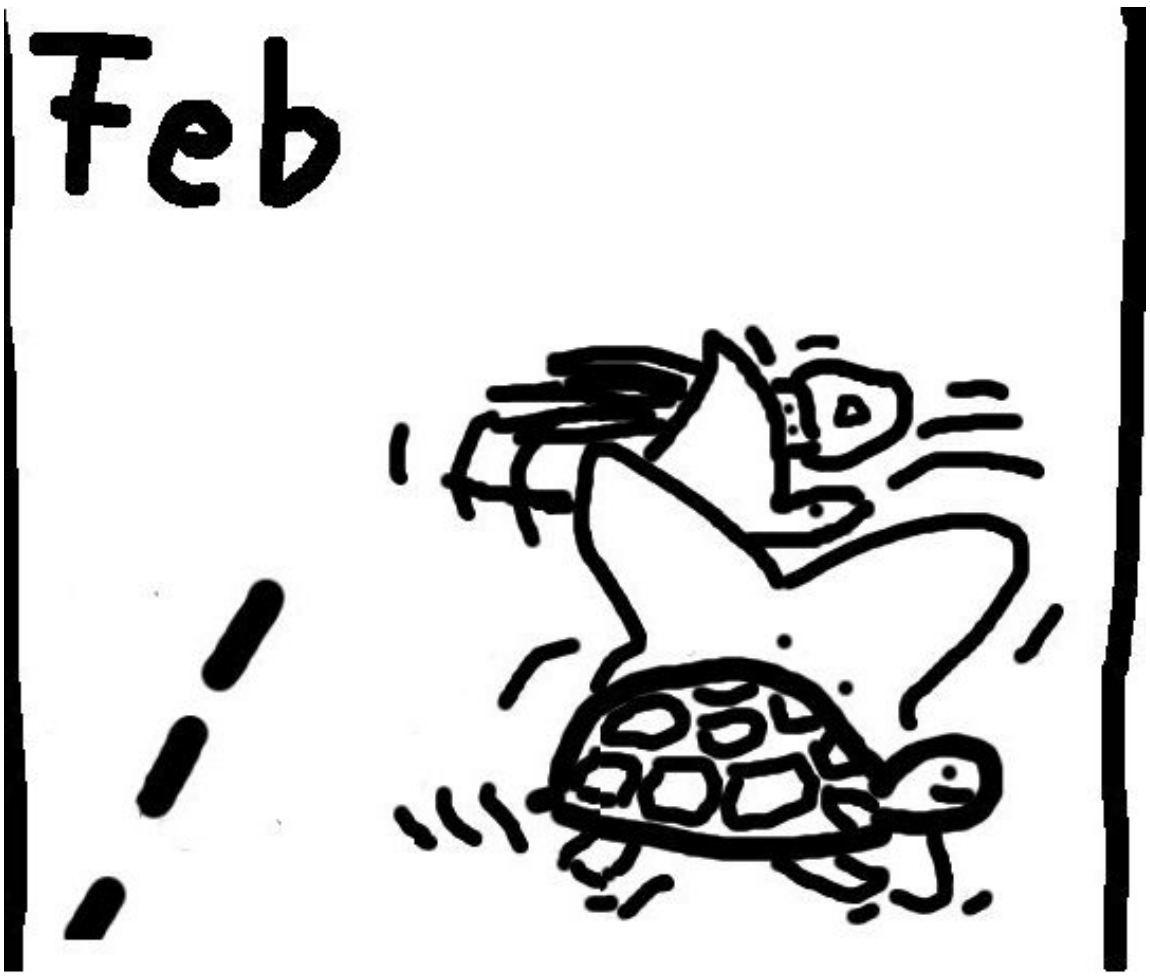



START

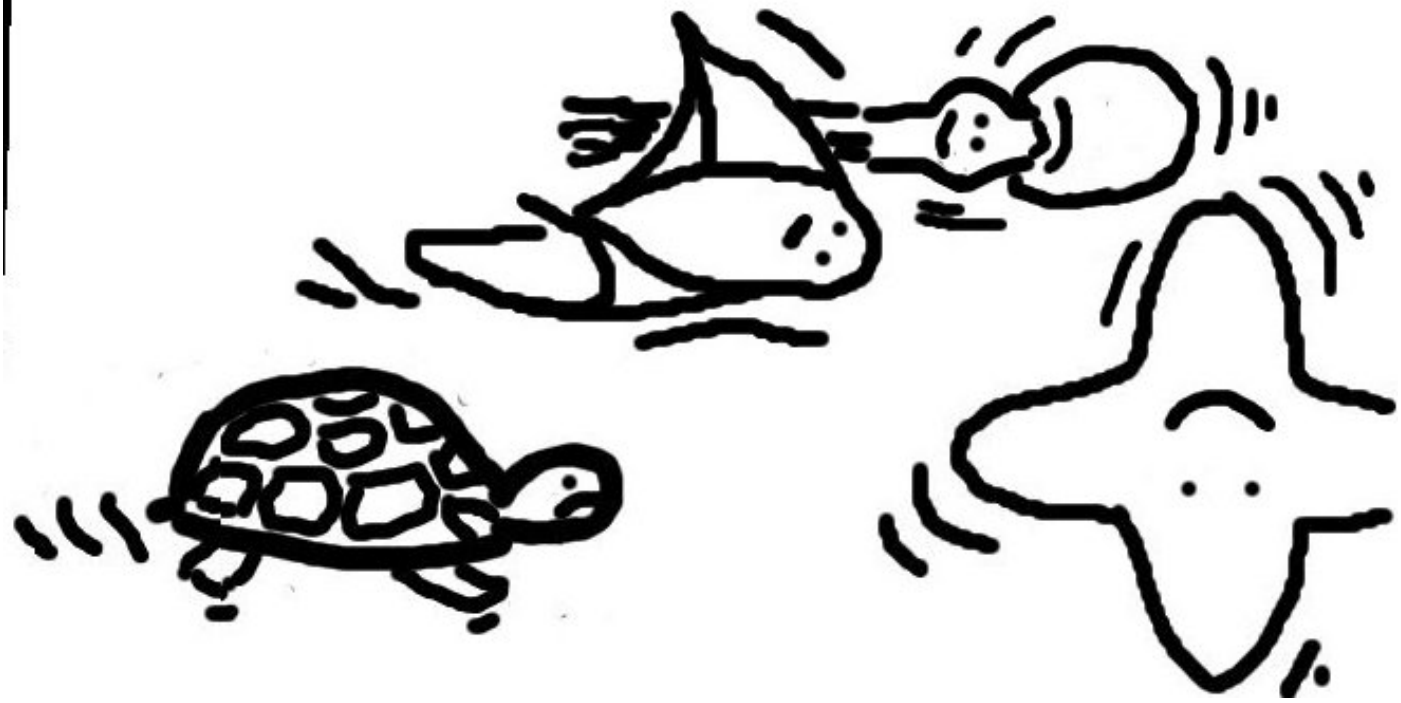

1985

2007-Jan

Feb



Mar



More addition formulas

Explicit-Formulas Database:
hyperelliptic.org/EFD

EFD has 581 computer-verified formulas and operation counts for ADD, DBL, etc.
in 51 representations
on 13 shapes of elliptic curves.

Not yet handled by computer:
generality of curve shapes
(e.g., Hessian order $\in 3\mathbf{Z}$);
complete addition algorithms
(e.g., checking for ∞).

Notation

If $$n = \sum_{i=0}^{l-1} n_i 2^i$$

we write n in binary representation

$$n = (n_{l-1} \dots n_0)_2.$$

E.g. $n = 35 = 32+2+1 = 1\cdot 2^5 + 0\cdot 2^4 + 0\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0$, then $35 = (100011)_2$.

The following algorithms are stated in some group $(G, +)$ with neutral element O. Scalar multiplication is denoted by $nP = P + P + \cdots + P$ (n terms).

Right–to–Left Binary

IN: An element $P \in G$ and a positive integer $n = (n_{l-1} \ldots n_0)_2$.

OUT: The element $nP \in G$.

$R \leftarrow O$, $Q \leftarrow P$,

for $i = 0$ to $l - 2$ do

    if $n_i = 1$ then $R \leftarrow P + Q$

    $Q \leftarrow 2Q$

if $n_{l-1} = 1$ then $R \leftarrow P + Q$

return R

This algorithm computes $35P = 2^5P + 2^1P + P$.

At the end of step 2, $Q = 2^{i+1}P$ and $R = (n_i \ldots n_0)_2 P$.

Left–to–Right Binary

IN: An element $P \in G$
and a positive integer $n = (n_{l-1} \ldots n_0)_2, n_{l-1} = 1$.
OUT: The element $nP \in G$.
$R \leftarrow P$
for $i = l - 2$ to 0 do
    $R \leftarrow 2R$
    if $n_i = 1$ then $R \leftarrow P + R$
return R

This algorithm computes $35P = 2(2(2(2(2P))) + P) + P$.
The intermediate variable R holds $(n_{l-1} \ldots n_i)_2 P$.

Number of additions

For each 1 in the binary representation of n we compute an addition. On average there are $l/2$ non-zero coefficients.

In some groups (e.g. elliptic curves) $P + Q$ has the same cost as $P - Q$, so it makes sense to use negative coefficients. This gives signed binary expansions.

Note that $31 = 2^4 + 2^3 + 2^2 + 2 + 1 = 2^5 - 1$ and so $31P = 2(2(2(2P + P) + P) + P) + P)$

$$= 2(2(2(2(2P)))) - P$$

Can always replace two adjacent 1's in the binary expansion by $10\bar{1}$ since $(11)_2 = (10\bar{1})_s$.
($\bar{1}$ denotes -1).

By systematically replacing runs of 1's we can guarantee that there are no two adjacent bits that are nonzero.

A representation fulfilling this is called a "non-adjacent form" (NAF).

NAF's have the lowest density among all signed binary expansions (with coefficients in $\{0, 1, -1\}$).

$$(10010100110111010\underline{11}0)_2 =$$
$$(10010100110111\underline{11}0\bar{1}0)_2 =$$
$$(1001010011\underline{1111}0\bar{1}0\bar{1}0)_2 =$$
$$(100101001\underline{111}000\bar{1}0\bar{1}0\bar{1}0)_2 =$$
$$(1001010100\bar{1}000\bar{1}0\bar{1}0\bar{1}0)_2$$

Results no worse, but not necessarily better:
$35 = (100011)_2 = (10010\bar{1})_s$.

Non-Adjacent Form

IN: Positive integer $n = (n_l n_{l-1} \ldots n_0)_2$, $n_l = n_{l-1} = 0$.

OUT: NAF of n, $(n'_{l-1} \ldots n'_0)_s$.

$c_0 \leftarrow 0$

`for` $i = 0$ `to` $l - 1$ `do`

$c_{i+1} \leftarrow \lfloor (c_i + n_i + n_{i+1})/2 \rfloor$

$n'_i \leftarrow c_i + n_i - 2c_{i+1}$

`return` $(n'_{l-1} \ldots n'_0)_s$

Resulting signed binary expansion has length at most $l + 1$, so longer by at most 1 bit.

On average there are $l/3$ non-zero coefficients.

Can also do left-to-right.

NAF – example

$c_0 \leftarrow 0$

$\texttt{for } i = 0 \texttt{ to } l - 1 \texttt{ do}$

$\quad c_{i+1} \leftarrow \lfloor (c_i + n_i + n_{i+1})/2 \rfloor$

$\quad n'_i \leftarrow c_i + n_i - 2c_{i+1}$

$\texttt{return } (n'_{l-1} \ldots n'_0)_s$

$35 = (00100011)_2,\ c_0 = 0$

$c_1 = \lfloor (0 + 1 + 1)/2 \rfloor = 1,$

$n_0 = 0 + 1 - 2 = -1$

$c_2 = \lfloor (1 + 1 + 0)/2 \rfloor = 1,$

$n_1 = 1 + 1 - 2 = 0$

$c_3 = \lfloor (1 + 0 + 0)/2 \rfloor = 0,$

$n_2 = 1 + 0 - 0 = 1$

$$c_4 = \lfloor (0+0+0)/2 \rfloor = 0,$$
$$n_3 = 0+0-0 = 0$$

$$c_5 = \lfloor (0+0+1)/2 \rfloor = 0,$$
$$n_4 = 0+0-0 = 0$$

$$c_6 = \lfloor (0+1+0)/2 \rfloor = 0,$$
$$n_5 = 0+1-0 = 1$$

$$c_7 = \lfloor (0+0+0)/2 \rfloor = 0,$$
$$n_6 = 0+0-0 = 0$$

$$\Rightarrow 35 = (100010\bar{1})_s$$

Generalizations

So far all expansions in base 2 (signed or unsigned).

Generalize to larger base; often 2^w ($w > 1$). Then the coefficients are in $[0, 2^w - 1]$. Also fractional windows have been suggested.

w is called window width.

Assume that mP for $m \in [0, 2^w - 1]$ are precomputed.

Easiest way: just group w bits.

Sliding windows: Group w bits and skip forward if LSB is 0 (requires only odd integers in $[0, 2^w - 1]$) as coefficients and leads to $l/(w+1)$ additions).

If $-$ is cheap,
use signed sliding windows;
this leads to $l/(w+2)$ additions.

$(\underline{10}\,\underline{01}\,\underline{01}\,\underline{00}\,\underline{11}\,\underline{01}\,\underline{11}\,\underline{01}\,\underline{01}\,\underline{10})_2 =$
$(02\,01\,01\,00\,03\,01\,03\,01\,01\,02)_2 =$
$(2110313112)_4$,
needs 8 additions and
precomputed $2P$ and $3P$.

$(10010100\underline{11}01\,\underline{11}0110\underline{11}0)_2 =$
$(1001010003\,0103010030)_2,$
needs 7 additions and only
precomputed $3P$.

$(1001010011011101\underline{011}0)_2 =$
$(10010100\underline{111}0000\bar{3}0030)_2 =$
$(10010\underline{101}00\bar{1}0000\bar{3}0030)_2 =$
$(10\underline{011}00\bar{3}00\bar{1}0000\bar{3}0030)_2 =$
$(1000300\bar{3}00\bar{1}0000\bar{3}0030)_2$
needs 5 additions and
precomputed $3P$,
assuming that $-$ is available.

Montgomery Ladder

Consider the intermediate results (i is decreasing).

$Q_i = \sum_{j=i}^{l} n_j 2^{j-i} P$, put $R_i = Q_i + P$, then $Q_i = 2Q_{i+1} + n_i P = Q_{i+1} + R_{i+1} + n_i P - P = 2R_{i+1} + n_i P - 2P$.

This implies $(Q_i, R_i) = \begin{cases} (2Q_{i+1}, Q_{i+1} + R_{i+1}) & \text{if } n_i = 0 \\ (Q_{i+1} + R_{i+1}, 2R_{i+1}) & \text{if } n_i = 1 \end{cases}$

$13 = (1101)_2$:

$(Q_3, R_3) = (P, 2P)$

$(Q_2, R_2) = (3P, 4P)$

$(Q_1, R_1) = (6P, 7P)$

$(Q_0, R_0) = (13P, 14P)$

Idea of joint doublings

To compute

$n_1 P_1 + n_2 P_2 + \cdots + n_m P_m$

compute the doublings together,

i.e. write scalars n_i in binary:

$$
\begin{aligned}
n_1 &= n_{1,l-1}2^{l-1} + n_{1,l-2}2^{l-2} + \ldots + n_{1,0} \\
n_2 &= n_{2,l-1}2^{l-1} + n_{2,l-2}2^{l-2} + \ldots + n_{2,0} \\
&\vdots \quad = \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
n_m &= n_{m,l-1}2^{l-1} + n_{m,l-2}2^{l-2} + \ldots + n_{m,0}
\end{aligned}
$$

Idea of joint doublings

To compute

$n_1 P_1 + n_2 P_2 + \cdots + n_m P_m$

compute the doublings together,

i.e. write scalars n_i in binary:

$$
\begin{aligned}
n_1 &= n_{1,l-1}2^{l-1} + n_{1,l-2}2^{l-2} + \ldots + n_{1,0} \\
n_2 &= n_{2,l-1}2^{l-1} + n_{2,l-2}2^{l-2} + \ldots + n_{2,0} \\
&\vdots \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
n_m &= n_{m,l-1}2^{l-1} + n_{m,l-2}2^{l-2} + \ldots + n_{m,0}
\end{aligned}
$$

Compute as

$$
2\Big(2(n_{1,l-1}P_1 + n_{2,l-1}P_2 + \cdots + n_{m,l-1}P_m) + (n_{1,l-2}P_1 + n_{2,l-2}P_2 + \cdots + n_{m,l-2}P_m\Big) + \cdots
$$

etc. Even with precomputations, many more adds than doublings.

Applications

ECDSA verification uses 2 scalar multiplications ... just to add the results.

If base point P is fixed, precompute $R = 2^{l/2}P$ and include in the curve parameters. Split scalar $n = n_1 2^{l/2} + n_0$ and compute

$$n_1 R + n_0 P.$$

GLV curves split scalar in two halves to get faster scalar multiplication.

Verification in accelerated ECDSA can be extended to use 4 or even 6 scalars. Splitting of the scalar is done by LLL techniques.

Further applications in batch verification of signatures — many scalars — by taking random linear combinations.

Examples

$1338P + 1715Q =$
$(10100111010)_2P +$
$(11010110011)_2Q$ takes 20
doublings and 12 additions.

Given precomputed $P + Q$,
we can compute the same
in 10 doublings and 8 additions.

If $-$ is efficient
$(010101001\bar{0}10)_2P +$
$(100\bar{1}0\bar{1}0\bar{1}0\bar{1}0\bar{1})_2Q$ reduces
(compared to first line) the
number of additions to 10,
but needs 21 doublings.

Given precomputed
$P + Q$ and $P - Q$:
$(0101010\bar{0}\bar{1}010)_2 P +$
$(100\bar{1}0\bar{1}0\bar{1}0\bar{1}0\bar{1})_2 Q$
needs 11 doublings
and 8 additions.

The joint Hamming weight
has not decreased,
and length has increased.

Joint-Sparse-Form

Solinas introduced generalization of NAF to two scalars:

Of any three consecutive columns, at least one is a zero column, that is for all i and all positive $j > 1$ one has $n_{i,j+k} = n_{1-i,j+k} = 0$ for at least one k in $\{0, \pm 1\}$.

Adjacent terms do not have opposite signs, i.e. it is never the case that $n_{i,j} n_{i,j+1} = -1$.

If $n_{i,j+1} n_{i,j} \neq 0$ then one has $n_{1-i,j+1} = \pm 1$ and $n_{1-i,j} = 0$.

$$
\begin{aligned}
&1338 = \\
&(10100111010)_2 \quad = \\
&(0101010010\bar{1}010)_2 \\
&1715 = \\
&(11010110011)_2 \quad = \\
&(100\bar{1}0\bar{1}0\bar{1}00011)_2
\end{aligned}
$$

So $1338P + 1715Q$ needs 7 additions and 11 doublings given precomputed $P + Q$ and $P - Q$. Average joint density is $1/2$, i.e. for two integers of length l (each) we need on average l doublings and $l/2$ additions.

Joint sparse form recoding

IN: Nonnegative l-bit integers n_0 and n_1 not both zero.
OUT: The joint sparse form of n_0 and n_1.
$j \leftarrow 0$, $d_0 \leftarrow 0$ and $d_1 \leftarrow 0$
while $n_0 + d_0 > 0$ OR $n_1 + d_1 > 0$ do
    $l_0 \leftarrow d_0 + n_0$
    $l_1 \leftarrow d_1 + n_1$
    for $i = 0$ to 1 do
        if l_i is even
        then $n_{i,j} \leftarrow 0$
        else $n_{i,j} \leftarrow l_i$ mods 4
        if $l_i \equiv \pm 3 \pmod 8$

```
            AND l_{1-i} ≡ 2    (mod 4)
            then n_{i,j} ← −n_{i,j}
        for i = 0 to 1 do
            if 2d_i = 1 + n_{i,j}
            then d_i ← 1 − d_i
            n_i ← ⌊n_i/2⌋
        j ← j + 1
return (n_{0,l} ... n_{0,0})_s
and (n_{1,l} ... n_{1,0})_s
```

Asymptotic speeds

1939 Brauer algorithm:

$\approx (1 + 1/\lg\lg H) \lg H$
additions to compute
$P \mapsto nP$ if $n \leq H$.

1964 Straus algorithm:

$\approx (1 + k/\lg\lg H) \lg H$
additions to compute
$P_1, \ldots, P_k \mapsto n_1 P_1 + \cdots + n_k P_k$
if $n_1, \ldots, n_k \leq H$.

1976 Yao algorithm:

$\approx (1 + k/\lg\lg H) \lg H$
additions to compute
$P \mapsto n_1 P, \ldots, n_k P$
if $n_1, \ldots, n_k \leq H$.

1976 Pippenger algorithm:

Similar asymptotics,
but replace $\lg\lg H$ with $\lg(k \lg H)$.
Faster than Straus and Yao
if k is large.
(Knuth summary is wrong.)

More generally, Pippenger's algorithm computes
ℓ sums of multiples of k inputs.

$$\approx \left(\min\{k, \ell\} + \frac{k\ell}{\lg(k\ell \lg H)} \right) \lg H$$

additions
if all coefficients are below H.
Within $1 + \epsilon$ of optimal.

More generally, Pippenger's algorithm computes
ℓ sums of multiples of k inputs.

$$\approx \left(\min\{k, \ell\} + \frac{k\ell}{\lg(k\ell \lg H)} \right) \lg H$$

additions
if all coefficients are below H.
Within $1 + \epsilon$ of optimal.

Various special cases of Pippenger's algorithm were reinvented and patented by 1993 Brickell–Gordon–McCurley–Wilson, 1995 Lim–Lee, etc.
Is that the end of the story?

No! 1989 Bos–Coster:

If $n_1 \geq n_2 \geq \cdots$ then
$n_1 P_1 + n_2 P_2 + n_3 P_3 + \cdots =$
$(n_1 - q n_2) P_1 + n_2 (q P_1 + P_2) +$
$n_3 P_3 + \cdots$ where $q = \lfloor n_1 / n_2 \rfloor$.

Remarkably simple;
competitive with Pippenger
for random choices of n_i's;
much better memory usage.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$010011010 = 154 \leftarrow$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $154P$, $146P$, $77P$, $2P$, $1P$.

Plus one extra addition:

add $146x$ into $154x$,

obtaining $300x$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$000001000 = 8 \leftarrow$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

plus 2 additions.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$000001000 = 8$

$001000101 = 69 \leftarrow$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

plus 3 additions.

Reduce largest row:

$$000100000 = 32$$
$$000010000 = 16$$
$$000001000 = 8$$
$$001000101 = 69$$
$$000001000 = 8 \leftarrow$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 4 additions.

Reduce largest row:

$$000100000 = 32$$
$$000010000 = 16$$
$$000001000 = 8$$
$$000100101 = 37 \leftarrow$$
$$000001000 = 8$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 5 additions.

Reduce largest row:

$$000100000 = 32$$
$$000010000 = 16$$
$$000001000 = 8$$
$$000000101 = 5 \leftarrow$$
$$000001000 = 8$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 6 additions.

Reduce largest row:

$000010000 = 16 \leftarrow$

$000010000 = 16$

$000001000 = 8$

$000000101 = 5$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 7 additions.

Reduce largest row:

$000000000 = 0$

$000010000 = 16$

$000001000 = 8$

$000000101 = 5$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 7 additions.

Reduce largest row:

$000000000 = 0$

$000001000 = 8 \leftarrow$

$000001000 = 8$

$000000101 = 5$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 8 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0 \leftarrow$$
$$000001000 = 8$$
$$000000101 = 5$$
$$000001000 = 8$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 8 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0 \leftarrow$$
$$000000101 = 5$$
$$000001000 = 8$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 8 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000101 = 5$$
$$000000011 = 3 \leftarrow$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 9 additions.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000010 = 2 \leftarrow$

$000000011 = 3$

$000000010 = 2$

$000000001 = 1$

plus 10 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000010 = 2$$
$$000000001 = 1 \leftarrow$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 11 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0 \leftarrow$$
$$000000001 = 1$$
$$000000010 = 2$$
$$000000001 = 1$$

plus 11 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000001 = 1$$
$$000000001 = 1 \leftarrow$$
$$000000001 = 1$$

plus 12 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0 \leftarrow$$
$$000000001 = 1$$
$$000000001 = 1$$

plus 12 additions.

Reduce largest row:

$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0$$
$$000000000 = 0 \leftarrow$$
$$000000001 = 1$$

plus 12 additions.

Reduce largest row:

$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0 \leftarrow$

plus 12 additions.

Final addition chain: 1, 2, 3, 5, 8, 16, 32, 37, 69, 77, 146, 154, 300.

Short, no temporary storage, low two-operand complexity.