Algebra and discrete mathematics, homework sheet 4

Due: 20 March 2015, 13:45

You can hand in in groups of two or three. Please clearly write the names on the sheet.

1. Let $\phi : R_1 \to R_2$ be a ring homomorphism. Show that $\mathrm{Im}(\phi)$ is a subring of R_2.

2. Let $(\mathbb{C}, +, \cdot)$ denote the field of complex numbers with regular addition and multiplication. Let the sets M_1 and M_2 be defined as follows:

$$M_1 = \{a + b\sqrt[3]{6} + c\sqrt[3]{6}^2 | a, b, c \in \mathbb{Z}\} \subseteq \mathbb{C},$$

$$M_2 = \{a + b\sqrt{2} + c\sqrt{3} | a, b, c \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

   (a) Study whether $(M_1, \cdot)$ is a semigroup.
   (b) Study whether $(M_2, \cdot)$ is a semigroup.
   (c) Is $(M_1, +, \cdot)$ a subring of $(\mathbb{C}, +, \cdot)$? Why?

3. Consider the ring $R = \mathbb{Z}/2 \times \mathbb{Z}/7 \times \mathbb{Z}/9$.

   (a) How many elements does R have?
   (b) Compute the order of $R^\times$.
   (c) Compute the (multiplicative) order of $(1, 3, 4) \in R^\times$.
   (d) Does there exist an integer m such that $R \cong \mathbb{Z}/m$? If so, compute it, explain how to compute the ring homomorphism and the inverse of the ring homomorphism, and compute the image of $(1, 3, 2)$. If not, why not?
   (e) Find two elements $a, b \in R$ so that $a \cdot b = (0, 0, 0)$ but $a, b \neq (0, 0, 0)$.
   (f) Compute the number of elements $a \in R$ with $a \neq (0, 0, 0)$ for which there exists a $b \neq (0, 0, 0)$ such that $a \cdot b = (0, 0, 0)$.
   (g) How many such zero divisors exist in$R^\times$?

4. Let R be a ring and let $\phi : R \to R$ be a ring homomorphism. Show that the set

$$S = \{s \in R | \phi(s) = s\}$$

   is a subring of R.