Algebra and discrete mathematics, homework sheet 1

Due: 24 February 2015, 8:45

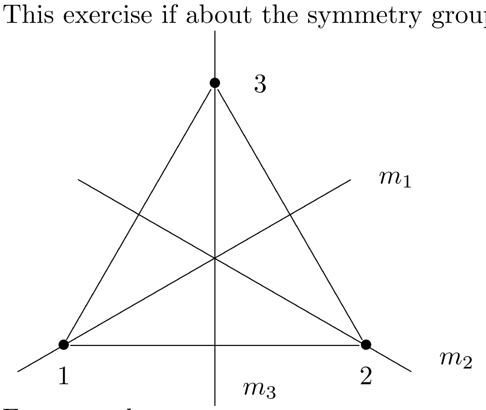
You can hand in in groups of three. We suggest you all try to solve the exercises by yourself and then consolidate your group's results into a single writeup that you hand in. Please clearly write the names and study number on all sheets.

1. Consider the subset $\mathbb{Z}[i]$ of the complex numbers given by

$$\mathbb{Z}[i] = \{a + bi \in \mathbb{C} | a, b \in \mathbb{Z}\}.$$

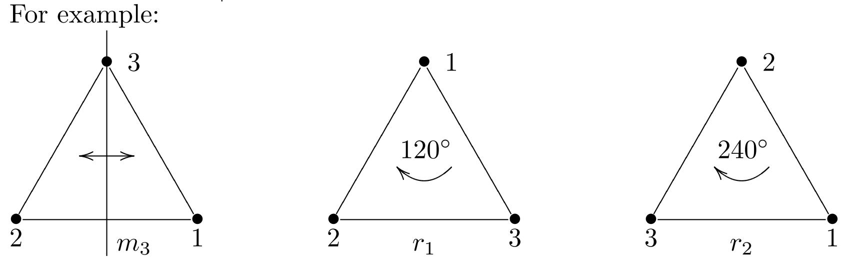
Show that $(\mathbb{Z}[i], +, 0)$ is a submonoid of the monoid $(\mathbb{C}, +, 0)$.

2. This exercise if about the symmetry group of the equilateral triangle.



Symmetry operations of the equilateral triangle are maps that do not change the shape of the triangle. There are 6 different such maps:
id: identity map,
m_1: reflection in axis through 1,
m_2: reflection in axis through 2,
m_3: reflection in axis through 3,
r_1: rotation by 120° mapping 1 to 3,
r_2: rotation by 240° mapping 1 to 2.

For example:



To determine whether the set of symmetry operations on the equilateral triangle forms a monoid with respect to composition first write a table with all results of composing two transformations. For maps we write $r_1 \circ m_1$ if first m_1 and then r_1 is executed. The table is to be read as follows: each table entry is the result of performing first the operation stated in same column in the top row, followed by the one in the same row in the leftmost column. E.g. $r_1 \circ m_1$ is found in the row of m_1 and the column of r_1 and equals m_2.

(a) Show that the set of symmetry operations on the equilateral triangle forms a monoid. To show this, write the complete multiplication table and identify the neutral element. You do not need to prove associativity.

(b) Find all submonoids.

3. Let $S := \{(a, b) \in \mathbb{Z}^2 | \ 2a + 3b \in 7\mathbb{Z}\}$.

(a) We define an operation $\circ$ on elements of S as follows:

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 + a_2, b_1 + b_2).$$

Find a candidate neutral element e and show that $(S, \circ, e)$ is a commutative monoid.

(b) We define a different operation $\diamond$ on S as follows:

$$(a_1, b_1) \diamond (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2).$$

Investigate whether $(S, \diamond, f)$ forms a monoid for some neutral element f.